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AN

EXERCISE BOOK IN ALGEBRA.

DESIGNED FOR

SUPPLEMENTARY OR REVIEW WORK

IN CONNECTION WITH

ANY TEXT-BOOK ON ALGEBRA.

BY

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BOSTON, U.S.A.
D. C. HEATH & CO., PUBLISHERS
1909

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PREFACE.

THIS book is designed to furnish a collection of exercises similar in character to those in the ordinary text-books, of medium grade as to difficulty, and selected with special reference to giving an opportunity for drill upon those subjects which experience has shown to be difficult for students to master.

Though intended primarily to be supplementary to some regular text-book, a number of definitions and a few rules have been added, in the hope that it may also be found useful as an independent review and drill book.

In general the common order of subjects has been followed, except that all the exercises of a kind have been put under one head.

These exercises have been (1) made up according to need, (2) adapted from forms in common use, and (3) copied without change from foreign text-books.

I am under obligation to PROF. WEBSTER WELLS, of the Institute of Technology, Boston, and to MR. GEO. T. EATON, Instructor in Mathematics, Phillips Academy, Andover, for suggestions and other assistance in the preparation of this book.

MATTHEW S. McCURDY.

ANDOVER, MASS., *Aug.*, 1894.

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EXERCISE BOOK IN ALGEBRA.

I. DEFINITIONS.

1. **Algebra** is the science which treats of numbers by means of symbols used to represent them, and to indicate their relations to each other and the operations to be performed upon them.

SYMBOLS OF QUANTITY.

2. **Known Quantities** are usually represented by the first letters of the alphabet, or by figures.

3. **Unknown Quantities** are usually represented by the last letters of the alphabet.

4. Quantities are *numerical* or *literal,* according as they are expressed by numbers or letters.

5. **Infinity,** or that which is greater than any assignable quantity, is represented by the symbol ∞.

6. **Zero,** denoting absence of value, or that which is smaller than any assignable quantity, is represented by the symbol 0.

SYMBOLS OF OPERATION.

7. **Addition** and **Subtraction** are indicated by the signs $+$ and $-$.

8. **Terms** are the quantities connected by these signs.

9. A **Coefficient** is a number or letter placed before a quantity to show how many times it is to be taken *as a term.*

10. **Multiplication** is indicated by the sign ×, or by a period; as, $5 \cdot 6 \cdot 7$, or by writing the quantities together without any sign; as, $2xyz$.

11. **Division** is indicated by the sign ÷, or the forms $\frac{a}{b}$, $a : b$.

12. The **Factors** of a quantity are the quantities that will produce it when multiplied together.

13. An **Exponent** is a figure or letter placed above, and a little to the right of, a quantity.

If the exponent is a *positive integral number*, it shows how many times the quantity is to be taken as a factor.

If it is a *positive fraction*, the denominator shows how many equal factors the quantity is to be divided into, and the numerator how many of these are to be used for a product.

If the exponent is *negative*, it indicates the reciprocal of what it would mean if positive.

14. The **Reciprocal** of a quantity is unity divided by the quantity; thus, a and $\frac{1}{a}$ are reciprocals of each other.

15. A **Power** is the product of any number of *equal* factors.

16. A **Root** is one of the equal factors of a quantity.

17. **Involution** is the process of producing powers from equal factors.

18. **Evolution** is the process of resolving a quantity into equal factors.

19. The **Radical Sign**, $\sqrt{}$, is used to indicate the square root of the quantity that follows it.

20. To indicate any other than the square root, a figure called the *index* is placed over the radical sign.

SYMBOLS OF RELATION.

21. The **Sign of Equality** is two parallel horizontal lines, $=$.

22. The **Signs of Inequality** are $>$ or $<$. $A > 6$ and < 10, is read A is greater than six and less than ten.

23. Geometrical Ratio is indicated by two dots in the form of a colon; as, $a : ar$.

24. The **Sign of Variation** is $\propto$; as, $x \propto y$, read x varies as y, and means that the ratio of x to y is a constant quantity.

SYMBOLS OF AGGREGATION.

25. (*a*) The **Vinculum,** $\overline{2a + b - c}$.

(*b*) **Parenthesis,** $(2x - y + z)$.

(*c*) **Brackets,** $[5x + zy - z]$.

(*d*) **Braces,** $\left\{2 - a - b\right\}$. (*e*) The **Bar,** $\begin{array}{r|l} A & x \\ -\,C & . \end{array}$

These are used to enclose two or more terms when they are to be taken as a single quantity, and are often referred to collectively as *parentheses* or *brackets.*

OTHER SYMBOLS AND DEFINITIONS.

26. Signs of Continuation are a series of dots, ..., or dashes, — — — —, and show that a series of numbers is to be continued in like manner to some particular term, or without limit.

27. Signs of Deduction are $\therefore$, *therefore,* and $\because$, *since* or *because.*

28. An **Algebraic Expression** is any quantity expressed in algebraic symbols.

29. A **Monomial** is an algebraic expression having only one term.

30. A **Binomial** has two terms.

31. A **Trinomial** has three terms.

32. A **Polynomial** has two or more terms.

33. The **Degree** of a term is the sum of the exponents of its literal factors.

34. **Homogeneous Terms** are those of the same degree.

35. **Similar Terms** are those containing the same letters with the same exponents.

36. **Dissimilar Terms** contain different letters, or the same letters with different exponents.

37. The terms **Positive** and **Negative,** indicated by the signs + and —, denote opposition, or reversal of direction; as, when motion north is called positive, motion south would be negative. In Algebra + and — usually denote addition and subtraction, except when used with exponents, then they denote multiplication and division.

II. NUMERICAL SUBSTITUTION.

38. If $a = 8$, $b = 1$, $c = 2$, $d = 3$, $x = 4$, $y = 5$, $z = 6$, find the numerical value of:

1. $4\,b^2 - 3\,c^2 + d^2$.
2. $\dfrac{4\,b^2 - a^2}{2\,cd}$.
3. $\dfrac{3\,x^2 - y^2}{4\,y + d} + 2\,ax - 3\,ay$.
4. $\left(\dfrac{b}{c}\right)^4 + \left(\dfrac{c+d}{y^2}\right)^3 - \dfrac{d-c}{y^3}$.
5. $4\,(x^2 - a^2) - 2\,(y^2 - c^2) - 3\,(z^2 - d^2)$.
6. $a^2\,(b^2 + c^2) + b^2\,(a^2 + c^2) + c^2\,(a^2 + b^2)$.
7. $4\,(b + c)\,(b - c)^2 - 3\,(x + y)\,(x - y)^2$.
8. $(b + c)^2 + b\,(c + d)^2$.
9. $(3\,a + b)\,(3\,a^2 - b^2) - (2\,x + y)\,(2\,x^2 - y^2)$.
10. $\sqrt{y^2 - x^2} + \sqrt[3]{z^2 - d^2} + \sqrt[4]{x^3 + c^2 + d^2 + 4b}$.
11. $\sqrt[3]{3\,ayz^2 + 2\,c^2y^2 + 4\,b^3x - 22\,xy}$.
12. $\sqrt[5]{d^4 - c^2x^2 + b^2dy}$.

III. ADDITION.

NUMERICAL COEFFICIENTS.

Add the following:

39. 1. $4a^2 + 3b - d + c,\ c - d - b + a^2,$
$2d - c + 2a^2 - 2b,\ d - e - c + b + 2a^2,$
$a^2 - b - c - d.$

2. $ab + c + b,\ 2ab - c + 2b,\ 2b - d^2 - c - ab,$
$2c - 3ab + b + d^2 + 3bc.$

3. $2b^3 - a^2b - 15a^3,\ 9b^3 - 7ab^2 - 11a^2b - 4a^3,$
$6a^3 - 4ab^2 + 9b^3,\ 6a^2b - 2a^3,\ 12a^2b - b^3.$

4. $a^3 + 3a^2b + 3ab^2 + b^3,\ -5ab^2 + 3a^2b - b^3 + 3a^3,$
$3ab^2 - 5a^2b + 3b^3 - 3a^3,\ -5b^3 + 2a^2b - 4a^2$
$+ 3ab^2,\ 7a^3 + 6b^3 - 5a^2b + 5ab^2,\ 4a^2b + 3ab^2$
$- 3b^3 + 4a^3.$

5. $a^5 + 5a^4b + 6a^2b^2c - 7ab,$
$6a^5 - a^4b - 6a^2b^2c + 10ab,$
$-2a^5 + 4a^4b + 12a^2b^2c - 10ab,$
$5a^5 - 16a^4b - 11a^2b^2c + 13ab,$
$-10a^5 + 8a^4b + a^2b^2c - 6ab.$

6. $a^3 - 2b^3 - 11a^2b - 4ab^2 - bc^2,$
$4b^3 - abc + 6c^3 + 9a^2b,$
$-ac^2 + ab^2 + c^2 - 2b^3 - a^3 + 4bc^2 + 3ab^2,$
$2a^2b - 3bc^2 - 7c^3 + 3abc.$

7. $2a + \sqrt{x} + \sqrt{y} + z,\ 2z - 3a - 2\sqrt{x} - 3\sqrt{y},$
$2a - 3\sqrt{x} - \sqrt{y} + 3z - v,$
$5a - \sqrt{x} - \sqrt{y} + z + v,$
$5\sqrt{x} - 6a + 4\sqrt{y} - z + 2v.$

8. $2\sqrt{x-y} - 3x + 3\sqrt{x}, 3\sqrt{x-y} - x - \sqrt{x},$
$2x + \sqrt{x} - \sqrt{x-y}, 3x - \sqrt{x-y} - 2\sqrt{x}.$

9. $6x^2y + \frac{1}{2}x^3y^2 - \frac{1}{3}x^4y^4, -7x^2y - \frac{1}{4}x^3y^2 + \frac{1}{2}x^4y^4,$
$2x^2y + \frac{1}{8}x^3y^2 - \frac{1}{8}x^4y^4,$
$-9x^2y + \frac{1}{16}x^3y^2 - \frac{3}{5}x^4y^4.$

10. $\frac{1}{2}x + \frac{1}{3}a^2b^2 - \frac{1}{4}xyz + \frac{1}{5}(a-b),$
$\frac{1}{3}x + \frac{1}{2}a^2b^2 - \frac{1}{8}xyz - \frac{1}{10}(a-b),$
$\frac{3}{4}x + \frac{5}{6}a^2b^2 + \frac{7}{12}xyz - \frac{1}{15}(a-b),$
$-\frac{1}{4}x + \frac{7}{12}a^2b^2 + \frac{3}{4}xyz - \frac{3}{5}(a-b).$

11. $3x^{\frac{1}{2}}y - 2x\sqrt{y} + 7, -2x^{\frac{1}{2}}y + 2x\sqrt{y} - 3,$
$4x^{\frac{1}{2}}y - x\sqrt{y} + 2, 3x^{\frac{1}{2}}y - 2x\sqrt{y} + 2.$

40. Literal Terms similar with respect to part of their factors.

12. $amx + ady - ac + ax, mbx - bdyz + bx.$

13. $aby + 2ax - 3bx + axy + 2y.$

14. $3x^4y^2 + x^{-m}y^{\frac{3}{2}} - 2x^8, -5sx^4y^2 + 2r^2x^{-m}y^{\frac{3}{2}} + 6x^8,$
$2a + x^4y^2 - b^2x^{-m}y^{\frac{3}{2}}, + 3a - 2x^8 + cy.$

15. $2x^{\frac{1}{2}}y^{\frac{1}{2}} - 3x^{-1} + 2ba^{-m}b^{-m},$
$2ax^{\frac{1}{2}}y^{\frac{1}{2}} - 2x^{-1} + 2ba^{-m}b^{-m},$
$x^{\frac{1}{2}}y^{\frac{1}{2}} - 3ax^{-1} + bxa^{-m}b^{-m}.$

16. $(a+b-c)x + (a-b-c)y + (b+c-a)x$
$+ (b-c-a)y + (c+a-b)x + (c-a-b)y$
$+ (a+b+c)y.$

17. $ax + x^2y + c + y, 2x^2y + 3axy - ac - ay,$
$3ax - bx^2y - bc - y,$
$2y + 3axy - x^2yz - c + 2y.$

41. Expressions having similar compound factors.

18. $2\sqrt{x-y} + a\sqrt{x+y} + b\sqrt{x},$
$3\sqrt{x-y} + 2a\sqrt{x+y} + c\sqrt{x},$
$a\sqrt{x} - 3\sqrt{x-y} - 3\sqrt{x+y}$
$-5\sqrt{x-y} - b\sqrt{x+y}.$

19. $2(a+b-c) + a\sqrt{a-m} + b\sqrt{m+n},$
$-(a+b-c) + 3b\sqrt{a-m} + \sqrt{m+n},$
$b\sqrt{a-m} - (a+b-c) + a\sqrt{m+n},$
$-a\sqrt{a-m} - 2b\sqrt{m+n} - (a+b-c).$

20. $(a+b+c)(x+a) + (b-c)(x+a)$
$+ (c+d)(x+a).$

21. $(2a-b+c)\sqrt{x^2-y^2} + (3a+b-c)\sqrt{x^2-y^2}$
$+ (2a-b-c)\sqrt{x^2-y^2}.$

22. $(\sqrt{a-b}+x)\sqrt{a+b} + (2\sqrt{a-b}+c)\sqrt{a+b}$
$+ (x-c-3\sqrt{a-b})\sqrt{a+b}.$

IV. SUBTRACTION.

SIMPLE SUBTRACTION.

42. 1. From $6a^2 + 7b^2 + 7c^2 + 6bc$
take $4a^2 - 6b^2 + 8bc - 6ab + 6c^2$.

2. From $4x^3 + z^3 + x^2y + 3xz^2 + 12xyz$
take $3x^3 - 11y^3 + 3z^3 - 2x^2y - 4xz^2 + 16xyz$.

3. From $11x^4 - 2x^3 + 3x^2 - 8x$
take $3x^4 - x^2 + 7x - 14$.

4. From $4a^3b + 7a^2b^2 - 2b^4$
take $2a^4 - 7a^2b^2 - 2ab^3 + 3b^4$.

5. From $x^3 + x^2 + 64$ take $x^3 - 4x^2 + 15x + 2$.

6. From $3x^4 - 2x^3 - x^2 + x + 1$
take $x^4 - 2x^3 + 2x^2 + 2x + 1$.

7. From $12x^5 + 5x$ take the sum of
$4x^4 - 2x^3 + 3x^2 + 9$
and $6x^5 + 2x^4 + 2x^3 + 5x - 3$.

8. From $10a^4 - 4a^3x - 2a^2x^2 - ax^3$ take the sum of
$a^4 - 2a^3x + x^4$, $2a^4 + 5a^3x + 3a^2x^2 - 5x^4$,
and $6a^4 + a^3x + a^2x^2 - 4ax^3$.

9. From the sum of $12x^5 + 4xy^4 + y^5$,
$2x^5 - 4x^4y - xy^4 + 3y^5$,
and $6x^4y + 2x^2y^3 - 3xy^4$, take the sum of
$6x^5 + 2x^2y^3 - y^5$, $x^5 - 2x^4y + x^3y^2 + 2y^5$,
and $6x^5 + 4x^4y - 2x^3y^3 + 3y^5$.

10. From the sum of $3x^2y^2 - xy^3 - y^4$,
$2x^2y^2 - xy^3 + 2y^4$, $x^2y^2 + 3xy^3 + y^4$ take the sum
of $2x^2y^2 + 3xy^3 + y^4$, $-3x^2y^2 - 2xy^3 + y^4$,
$-x^2y^2 - xy^3 - y^4$.

11. From $(x + y + z)\,a + (2x + y)\,a$ take $(x - z)\,a$.

12. From $(3a + b - c)(x + y) + (a - b)(x + y)$
take $(a - b - c)(x + y)$.

13. From $(2a - 3x)\,a^2b^2 + (a + x)\,a^2b^2 - 3a^2b^2x$
take $(a - x)\,a^2b^2$.

REMOVAL OF PARENTHESES.

43. Remove the following parentheses and combine the terms.

14. $2a^2 - (3a^2 + 2b + c) + a^2$
$- (2b - c) + 2b - (3a^2 + 3b + c)$.

15. $3x - (2a + x - \overline{a + b}) + 3a$
$- (2x - a - b) + 2b - [a - b - (x + a)]$.

16. $2a - (a - b - \overline{c + b}) - b + b$
$- [a - c - (2c - b)] - 2b$
$- [a + b - \overline{a + c} + b]$.

17. $x^4 - [4x^3 - \{6x^2 - (4x - 1)\}]$
$- (x^4 + 4x^3 + 6x^2 + 4x + 1)$.

18. $a - [5b - \{a - (5c - \overline{2c - b} - 4b) + 2a$
$- (a - \overline{2b + c})\}]$.

19. $2a - (3b + 2c) - [5b - (6c - 6b) + 5c$
$- \{2a - (c + 2b)\}]$.

20. $15x - \{4 - [3 - 5x - (3x - 7)]\}$.

21. $2x - (5y - \overline{3z + 7})$
$- [4 + \{x - (3y + 2z + 5)\}]$.

22. $a - [3b + \{3c - (d - b) + a\} - 2a]$.

23. $2a - \{3b + (4c - \overline{3b + 2a})\}$.

24. $2a - 3(a - \overline{b - c}) - [a - (b - \overline{a + b} + c)$
$+ d - 2a] - (b - c).$

25. $-10(a - 6[a - \overline{b - c}]) + 60[b - (c + a)].$

26. $-2[a - \{a + b - \overline{a - b} - b\}$
$- \{2a - (3b + a - \overline{4a - b})\}].$

27. $a - 2(b - c) - [- \{- (4a - b - c$
$- 2\{a + b + c\})\}].$

28. $-5\{8a - 2b - (c + a) - [- (a + b - c)$
$+ 2a - b]\}.$

29. $-2\{- [- (2x - y)]\} + \{- 4(- [x - 2y])\}.$

30. $a - [a - \overline{2 - 3a} - (2b - \overline{4a + 2b})]$
$+ [3a - \{6a - (2a - 3b)\}].$

31. $1 - [1 - \overline{1 - 4x} - \{2x - \overline{3 - 5x}\}]$
$- \{2 - (5x - 4)\}.$

32. $2x - [3y + (2y - z) - 4z$
$+ \{2x - (3y - \overline{z - 2y})\}].$

33. $4(1 + 2x) - 2(3x + 2[2x - \overline{4 - 1}]).$

34. $84 - 7[- 11x - 4\{- 17x + 3(8 - \overline{9 - 5x})\}].$

35. $x^4 - [3x^3 + \{2x^2 - (x + 1)\}]$
$+ \{- x^4 - (- \overline{x + 7} + 3x^2)\}.$

36. $x^3 - \{x^2 - (x + 1)\} - [x^3 - \{x^2 - (x + 1)\}].$

37. $1 - 2x - \{3 - (4 - 5x)\} + \{6 - (7 - 8x)\}.$

38. $\frac{1}{4}(a - 5[b - a]) - \frac{3}{2}\left[\frac{1}{3}\left(b - \frac{a}{3}\right)\right.$
$\left. - \frac{2}{9}\left\{a - \frac{3}{4}\left(b - \frac{4a}{5}\right)\right\}\right].$

39. $2 - \frac{1}{4}(a + 3) + \frac{1}{2}\{5 - (6 - 3a)\}$
$+ \frac{1}{4}\{2a - (3a - 1)\}.$

40. $(a - b)(b - c)(c - a) - [a^2(c - b)$
$- \{c^2(a - b) - b^2(c + a)\}].$

41. $x - [y - (x - y) - \{x - (y - x) - y\}$
$- \{x - y - \overline{x - y} - x\}].$

42. $-20(a - d) + 3(b - c)$
$- 2[b + c + d - 3\{c + d - 4(d - a)\}].$

43. $35\left[\frac{3x - 4y}{5} - \frac{1}{10}\left\{3x - \frac{5}{7}(7x - 4y)\right\}\right]$
$+ 8(y - 2x).$

44. $\frac{3}{8}\left\{\frac{4}{3}(a - b) - 8(b - c)\right\} - \left\{\frac{b - c}{2} - \frac{c - a}{3}\right\}$
$- \frac{1}{2}\left\{c - a - \frac{2}{3}(a - b)\right\}.$

44. Bracket like powers of x, placing a minus sign before each compound expression.

45. $b - 2ax^3 - bx^3 - ax - cx^2$
$+ bx + cx^3 - 3ax^2 + bx^2 - cx - a.$

46. $c - ax - ax^2 + cx - bx^2 - bx + cx^2 - a - b.$

47. $x^4 - 2x^3 - ax^2 - 2ax^4 - 3ax^3 - bx^2 - x - a - 1.$

48. $x^2\sqrt{a + 1} - 3ax - \sqrt{a + 1} - 2ax^2 + x\sqrt{2} + \sqrt{2}.$

49. $ax - 2x + bx^2 - 2x^2 - x^3 + cx^3.$

V. MULTIPLICATION.

QUESTIONS AND EXERCISES.

45. **1.** What is the rule of signs?

2. What is the rule of exponents when multiplying like letters together?

3. Prove these two rules.

4. Give the general rule for multiplying two simple expressions together; a compound by a simple; two compound expressions.

Multiply the following:

5. $x^2 + ax + a^2$ by $x - a$.

6. $2x^2 - 4bx + 3b^2$ by $2x - 3b$.

7. $4x^3 - 3x^2 - 2x + 5$ by $3x - 5$.

8. $3a^3 + 5a^2 - 7a + 3$ by $2a - 3$.

9. $5a^3 - 3a^2b - 5ab^2 + 4b^3$ by $3a + 5b$.

10. $2y^3 + 3by^2 - 2b^2y - 3b^3$ by $4y - 3b$.

11. $x^4 - ax^3 + a^2x^2 - a^3x + a^4$ by $x + a$.

12. $3x^4 - 2x^3y + 4x^2y^2 - 7xy^3 + 16y^4$ by $2x - 3y$.

13. $1 + 4x - 10x^2$ by $1 - 6x + 3x^2$.

14. $x^3 - 4x^2 + 11x - 24$ by $x^2 + 4x + 5$.

15. $x^3 + 4x^2 + 5x - 24$ by $x^2 - 4x + 11$.

16. $x^3 - 7x^2 + 5x + 1$ by $2x^2 - 4x + 1$.

17. $x^3 + 6x^2 + 24x + 60$ by $x^3 - 6x^2 + 12x + 12$.

18. $x^3 - 2x^2 + 3x - 4$ by $4x^3 + 3x^2 + 2x + 1$.

19. $x^4 - 2x^3 + 3x^2 - 2x + 1$.
by $x^4 + 2x^3 + 3x^2 + 2x + 1$.

20. $a^2 + b^2 + ab - 3a + 3b + 9$ by $a - b + 3$.

21. $4x^2 + 9y^2 + z^2 - 6xy - 2xz - 3yz$ by $2x + 3y + z$.

22. $x^2 + 4y^2 + 3z^2$ by $x^2 - 2y^2 - 3z^2$.

23. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.

24. $a^2 + b^2 + c^2 + ab + ac - bc$ by $a - b - c$.

25. $a^3 + 2a^2b + 4ab^2 + 8b^3$ by $a^2 - 4ab + 4b^2$.

26. $3x^4 - 2x^3y + 4x^2y^2 - 7xy^3 + 16y^4$ by $2x - 3y$.

27. $x^5 - 2x^2 + 3$ by $2x^2 - 3x - 1$.

NEGATIVE AND FRACTIONAL EXPONENTS.

46. Follow the same rule as with positive integral exponents. The exponent of the product of $a^5 \times a^{-3}$ is 2, the algebraic sum of 5 and -3, the exponents of the factors.

EXERCISES.

1. What is the product of: $a^5 \times a^{-1}$, $3a^2b \times 2a^{-1}b^{-2}$,
$a^2b^2c^2 \times a^{-1}b^{-4}c^3$, $5a^{-1}b \times 5a^2b^{-2}$,
$2a^{\frac{1}{2}}b^{\frac{1}{2}} \times a^{\frac{1}{2}}b^{\frac{1}{2}}$, $a^m \times a^n$, $a^{m+1}b \times a^{m-1}b$.

2. Square: a^2, a^{-2}, $a^{\frac{1}{2}}$, $2a^{-1}b$, $a^{\frac{1}{4}}$, a^m, a^{2m}, $ab^{-1}c^{-1}$.

3. Cube: a, $2b^2$, $3ab$, $2abc^2$, $a^{-1}b^{-1}$, a^mb^m, $(a + b)^3$.

Multiply the following:

4. $x^{-2} + 2x^{-1} + 3$ by $x^{-2} + x^{-1} - 2$.

5. $x^{\frac{1}{2}} + x + x^{\frac{3}{2}} - 1$ by $x^{\frac{1}{2}} - x - x^{\frac{3}{2}} + 1$.

6. $x^{\frac{3}{2}} - 2x + x^{\frac{1}{2}} + x^{-\frac{1}{2}}$
by $x^{-\frac{3}{2}} - x^{-1} + x^{-\frac{1}{2}} + x.^{\frac{1}{2}}$ ($x^0 = 1$).

7. $m^{\frac{2}{3}} - 2a^{\frac{1}{4}}m^{\frac{1}{3}} + 4a^{\frac{1}{2}}$ by $m^{\frac{2}{3}} + 2a^{\frac{1}{4}}m^{\frac{1}{3}} + 4a^{\frac{1}{2}}$.

8. $x^{\frac{2}{3}} - 4 + 4x^{-\frac{2}{3}}$ by $x^{\frac{2}{3}} + 4 + 4x^{-\frac{2}{3}}$.

9. $a^{-2} + a^2$ by $a^{-2} - a^2$.

10. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

11. $x^{\frac{4}{5}} + x^{\frac{3}{5}}y^{\frac{1}{5}} + x^{\frac{2}{5}}y^{\frac{2}{5}} + x^{\frac{1}{5}}y^{\frac{3}{5}} + y^{\frac{4}{5}}$ by $x^{\frac{1}{5}} - y^{\frac{1}{5}}$.

12. $x^{-\frac{2}{3}} + x^{-\frac{1}{3}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}$ by $x^{-\frac{1}{3}} - y^{-\frac{1}{3}}$.

13. $x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y$ by $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

14. $x^{\frac{5}{2}} + x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}} + x^{-\frac{3}{2}}$ by $x^{\frac{1}{2}} - x^{-\frac{1}{2}}$.

THEOREMS IN MULTIPLICATION.

47. Two binomials having a common term.

1. Give the theorem for the product of two binomials having a common term.

2. Express this theorem by a formula derived from $(x + a)(x + b)$.

Find by inspection the following products:

3. $(x + 3)(x + 2)$.
4. $(x + 5)(x + 6)$.
5. $(x + 4)(x + 3)$
6. $(x + 12)(x + 1)$.
7. $(x + 5)(x + 1)$.
8. $(x - 3)(x - 5)$.
9. $(x - 2)(x + 3)$.
10. $(x - 4)(x + 1)$.
11. $(x - 3)(x - 5)$.
12. $(x^2 + 2)(x^2 + 1)$.
13. $(x^3 + 1)(x^3 - 2)$.
14. $(x^{-2} - 1)(x^{-2} + 1)$.
15. $(x^{\frac{1}{2}} + 6)(x^{\frac{1}{2}} - 5)$.
16. $(x^{-1} + 3)(x^{-1} + 2)$.
17. $(x^{\frac{1}{3}} + 5)(x^{\frac{1}{3}} - 1)$.
18. $(x^{-3} + 6)(x^{-3} + 2)$.

48. Three binomials having a common term.

1. What is the theorem for the product of three binomials having a common term?

2. Express this theorem by a formula derived from $(x + a)\ (x + b)\ (x + c)$.

Write by inspection the following products:

3. $(x + 2)\ (x + 3)\ (x + 4)$.
4. $(x + 3)\ (x + 5)\ (x + 1)$.
5. $(x + 7)\ (x + 2)\ (x + 6)$.
6. $(x - 1)\ (x - 2)\ (x - 3)$.
7. $(x + 2)\ (x - 1)\ (x + 5)$.
8. $(x^2 + 2)\ (x^2 + 3)\ (x^2 - 1)$.
9. $(x^3 + 6)\ (x^3 + 1)\ (x^3 + 2)$.
10. $(x - 6)\ (x - 2)\ (x - 3)$.
11. $(x^{-1} + 2)\ (x^{-1} + 3)\ (x^{-1} + 3)$.
12. $(x^{\frac{1}{2}} + 2)\ (x^{\frac{1}{2}} + 2)\ (x^{\frac{1}{2}} + 3)$.
13. $(x^{-2} + 1)\ (x^{-2} + 3)\ (x^{-2} - 3)$.
14. $(x^{\frac{2}{3}} + 1)\ (x^{\frac{2}{3}} + 2)\ (x^{\frac{2}{3}} + 3)$.

49. The square of the sum of two quantities.

1. Give the theorem and formula.

Expand by inspection:

2. $(a + 2)^2$.
3. $(a + 3)^2$.
4. $(a^2 + 1)^2$.
5. $(a + 5)^2$.
6. $(2a + 3x)^2$.
7. $(3x + 2)^2$.
8. $(2x^2 + y)^2$.
9. $(4x + 1)^2$.
10. $(3x + 2y^2)^2$.
11. $(x^{\frac{1}{2}} + 2y)^2$.
12. $(x^{-1} + 4y^{-1}).^2$
13. $(2x^{-2} + 5y^{-2}).^2$

50. The square of the difference of two quantities.

1. Give the theorem and formula.

Expand by inspection:

2. $(2a - 3x)^2$.
3. $(3a - 2x)^2$.
4. $(a^2 - b^2)^2$.
5. $(4a - 3x)^2$.
6. $(5x - y)^2$.
7. $(x - 2y)^2$.
8. $(5x^2 - 2x)^2$.
9. $(3x - 1)^2$.
10. $(a^{-1} - b^{-1})^2$.
11. $(a^{-2} - b)^2$.
12. $(2a^{-3} - 1)^2$.
13. $(x^{-1} - 2y^{-1})^2$.

51. The product of the sum and difference of two quantities.

1. Give the theorem and formula.

Find by inspection the following products:

2. $(a + b)(a - b)$.
3. $(a + 2)(a - 2)$.
4. $(2a - 3x)(2a + 3x)$.
5. $(5x + 2)(5x - 2)$.
6. $(3x - 6)(3x + 6)$.
7. $(5x - 2y)(5x + 2y)$.
8. $(x - 4)(x + 4)$.
9. $(3x - 4)(3x + 4)$.
10. $(2x - 3)(2x + 3)$.
11. $(8x - 1)(8x + 1)$.
12. $(x + 2)(x - 2)(x^2 + 4)$.
13. $(x - 3)(x + 3)(x^2 + 9)$.
14. $(x + y)^2 (x - y)^2$.
15. $(x^{-1} - y)(x^{-1} + y)(x^{-2} + y^2)$.
16. $(x^{-2} - 1)(x^{-2} + 1)(x^{-4} + 1)$.
17. $(x + y)(x - y)(x^2 + y^2)(x^4 + y^4)$.
18. $(y - 5)(y + 5)(y^2 + 25)(y^4 + 625)$.

52. The square of any polynomial.

1. Give the theorem, and write the formula obtained from $(a + b + c)^2$.

Expand by inspection:

2. $(2 - 3x - 4x^2)^2$.
3. $(x^2 + xy - y^2)^2$.
4. $(1 - 3x + 3x^2)^2$.
5. $(x^2 + 3x + 1)^2$.
6. $(3x^2 - 2x + 1)^2$.
7. $(4x^2 + 2x + 1)^2$.
8. $(2x^3 + x^2 - x - 1)^2$.
9. $(a^3 - a^2 - a - 1)^2$.
10. $(a^3 - \frac{a^2}{2} - a - 1)^2$.
11. $(3x^4 - 2x^3 + 3x^2 - x + 1)^2$.

53. The cube of any polynomial.

1. Give the theorem and obtain the formula from $(a + b + c)^3$.

Expand by inspection:

2. $(a + 2b)^3$.
3. $(2a + 3b)^3$.
4. $(2x^2 - 3)^3$.
5. $(2x - 5)^3$.
6. $(3x^2 + x + 1)^3$.
7. $(5x^2 - 3x - 2)^3$.
8. $(5a^3 + a^2 - 1)^3$.
9. $(x^3 - x^2 - x + 1)^3$.
10. $(2x^3 + 3x^2 - 2x + 1)^3$.

VI. DIVISION.

QUESTIONS AND EXERCISES.

54. 1. What is the rule of signs? Show how this follows from the rule in multiplication.

2. What is the rule of exponents when dividing like letters? Prove the rule.

3. Give the rule for dividing one simple expression by another; a compound by a simple; a compound by another compound expression. How should the terms of divisor and dividend be arranged?

Divide the following:

4. $x^3 + 3x^2 - 23x + 35$ by $x + 7$.

5. $x^3 - 12x^2 + 27x + 40$ by $x - 5$.

6. $2a^3 - 7a^2 - 3a + 18$ by $2a + 3$.

7. $3a^3 + 17a^2 - 43a + 20$ by $3a - 4$.

8. $3x^3 + x^2y - xy^2 + 4y^3$ by $3x + 4y$.

9. $x^4 - 8x^3y + 21x^2y^2 - 16xy^3 - 7y^4$
by $x^2 - 5xy + 7y^2$.

10. $x^4 - 9ax^3 + 12a^2x^2 + 35a^3x + 15a^4$
by $x^2 - 4ax - 3a^2$.

11. $4a^4 - 16a^3b - 4a^2b^2 + 40ab^3 + 25b^4$
by $2a^2 - 4ab - 5b^2$.

12. $4x^4 - 15ax^3 + 26a^2x^2 - 23a^3x + 8a^4$
by $4x^2 - 7ax + 8a^2$.

13. $5x^4 - 14x^3y + 31x^2y^2 - 22xy^3 + 12y^4$
by $5x^2 - 4xy + 3y^2$.

14. $2x^4 - 2x^3y - 5x^2y^2 + 4xy^3 + 5y^4$
by $2x^2 - 6xy + 5y^2$.

15. $a^5 - 5a^4b + 11a^3b^2 - 14a^2b^3 + 9ab^4 - 2b^5$
by $a^2 - 3ab + 2b^2$.

16. $a^2 + 4b^2 + 9c^2 - 12bc - 6ac + 4ab$
by $3c - 2b - a$.

17. $x^4 + 2x^2 + x + 2$ by $x^2 + x + 1$.

18. $x^4 + 2x^2 - x + 2$ by $x^2 - x + 1$.

19. $x^5 - 4x^4 + 3x^3 + 3x^2 - 3x + 2$ by $x^2 - x - 2$.

20. $a^6 + 2a^5 - 4a^4 - 2a^3 + 12a^2 - 2a - 1$
by $a^2 + 2a - 1$.

55. More difficult division.

Divide the following:

1. $x^2 - y^2 - z^2 + 2yz$ by $y - x - z$.
2. $x^2 - 4y^2 - z^2 - 4yz$ by $2y + z - x$.
3. $a^9 - b^9$ by $a^3 - b^3$.
4. $a^2b^2 + 2abc^2 - a^2c^2 - b^2c^2$ by $ab + ac - bc$.
5. $xy^3 + 2y^3z - xy^2z + xyz^2 - x^3y - 2yz^3 + x^3z - xz^3$
by $y + z - x$.
6. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
7. $x^3 + y^3 + 3xy - 1$ by $x + y - 1$.
8. $a^3 + b^3 - c^3 + 3abc$ by $a + b - c$.
9. $8x^3 - y^3 + z^3 + 6xyz$ by $y - z - 2x$.
10. $a^2(b + c) - b^2(a + c) + c^2(a + b) + abc$ by $a - b + c$.

56. Divide without clearing of parentheses.

1. $x^3 - (a + b + c)x^2 + (ab + bc + ac)x - abc$ by $x - a$.
2. $x^4 + (5 + a)x^3 - (4 - 5a + b)x^2 - (4a + 5b)x$
$+ 4b$ by $x^2 + 5x - 4$.
3. $(a + b)^3 + 4(a + b)^2 + 5(a + b) + 2$ by $a + b + 2$.

4. $(x^3 - 1)\,a^3 - (x^3 + x^2 - 2)\,a^2 + (4\,x^2 + 3\,x + 2)\,a$
$- 3\,(x + 1)$ by $(x - 1)\,a^2 - (x - 1)\,a + 3$.

5. $(a + b)^3 + 3\,(a + b)^2\,x + 3\,(a + b)\,x^2 + x^3$ by $a + b$
$+ x$.

6. $(x + y)^2 - 2\,(x + y)\,z + z^2$ by $x + y - z$.

7. $(x + y)^3 + 3\,(x + y)^2\,z + 3\,(x + y)\,z^2 + z^3$ by $(x + y)^2$
$+ 2\,(x + y)\,z + z^2$.

8. $x^3 + x^2\,(y + z) - xyz - y^2z - yz^2$ by $x^2 - yz$.

9. $ax^3 + (2\,a^2 + ac - b^2)\,x^2 + (2\,a^2c - 2\,ab^2 - b^2c)\,x$
$- 2\,ab^2c$ by $x + 2\,a$.

10. $a^{-6}\,b^m + a^{-5}\,(b^{m-2} - b^{m-1}) + a^{-4}\,(-2 - b^{m-3} + b^m)$
$- a^{-3}\,(2\,b^{-2} + b^{m-1}) - 2\,a^{-2} + a^{-1}\,b^2 + ab^2 + 1$ by
$a^{-4}\,b^{m-1} - a^{-3}\,b^{m-2} - 2\,a^{-2}\,b^{-1} + ab$.

57. Quotients that are infinite series.

Divide to five terms:

1. $2 - 3\,x + 4\,x^2$ by $1 + 2\,x - 5\,x^2$.
2. $1 - 3\,x^2$ by $2 - 3\,x - 2\,x^2$.
3. $a - 1$ by $1 - 2a + 2a^2$.
4. 1 by $1 - 2a + 4\,a^2$.
5. $1 + 3\,x$ by $1 - 2\,x$.
6. a by $1 + a$.

FRACTIONAL AND NEGATIVE EXPONENTS.

58. Use the same rule as with positive integral exponents.

1. Divide a^5 by a^{-2}, a^{-5} by a^2, $4\,a^2b^2$ by $2\,a^{-1}\,b^{-1}$, a^m by a^n,
a^{2m} by a^m, $b^{\frac{3}{2}}$ by $b^{\frac{1}{2}}$, a^4b^2 by $a^{-1}b^{-1}$.

2. Prove that any quantity with an exponent 0 is equal to unity.

3. Simplify $a^2a^{-2}a^{-4} \div a^{-6}$, $a^{\frac{1}{2}}a^{\frac{1}{4}}b^{\frac{2}{3}} \div a^{\frac{3}{4}}b^{\frac{2}{3}}$,
$a^2b^{-2}c^{\frac{2}{3}} \div a^{2m}b^{4m}c^{\frac{1}{2}}$.

4. Simplify $a^3b^{-3}c^{-6} \div a^{-3}b^{-9}c^{-6}$, $3^4a^2b^3c \div 27\,a^3b^3c^{-6}$.

Divide the following:

5. $x^{-2} - y^{-2}$ by $x^{-1} - y^{-1}$.

6. $x^2 - y^2$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

7. $x^{-3} - 8\,y^6$ by $x^{-1} - 2\,y^2$.

8. $x^3 + x^{-3}$ by $x + x^{-1}$.

9. $a - a^{\frac{1}{4}}b^{\frac{3}{2}}$ by $a^{\frac{1}{4}} - b^{\frac{1}{2}}$.

10. $x^2y^{-3} - xy^{-1} + x^{-1}y - xy^{-2} + x^{-2}y - x^{-3}y^2$ by $x^{-1}y^{-2} - x^{-2}y^{-1}$.

11. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

12. $2\,x^5y^{-3} - 5\,x^4y^{-2} + 7\,x^3y^{-1} - 5\,x^2 + 2\,xy$ by $x^3y^{-3} - x^2y^{-2} + xy^{-1}$.

13. $x - 5\,x^{\frac{2}{3}} - 46\,x^{\frac{1}{3}} - 40$ by $x^{\frac{1}{3}} + 4$.

14. $x - 4\,x^{\frac{3}{4}} + 6\,x^{\frac{1}{2}} - 4\,x^{\frac{1}{4}} + 1$ by $x^{\frac{1}{2}} - 2\,x^{\frac{1}{4}} + 1$.

SPECIAL THEOREM IN DIVISION.

59. 1. What is the theorem for the division of $a^n \pm b^n$ by $a \pm b$?

Divide the following:

2. $a^4 - b^4$ by $a - b$.

3. $a^3 - b^3$ by $a - b$.

4. $a^3 + b^3$ by $a + b$.

5. $a^6 - b^6$ by $a - b$.

6. $a^5 + b^5$ by $a + b$.

7. $a^6 + b^6$ by $a^2 + b^2$.

8. $a^{12} - b^{12}$ by $a^3 - b^3$.

9. $a^{12} + b^{12}$ by $a^4 + b^4$.

10. $27\,a^3 - 64\,b^3$ by $3\,a - 4\,b$.

11. $32\,a^5 + 243\,b^5$ by $2\,a + 3\,b$.

12. $x^{10} + y^{10}$ by $x^2 + y^2$.

13. $128\,a^{14} + 2187\,b^{21}$ by $2a^2 + 3\,b^3$.

14. $a^2 - b^2$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

15. $a + b$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

16. $a^{-4} - b^{-4}$ by $a^{-1} - b^{-1}$.

17. $1 - 32\,x^{-5}$ by $1 - 2\,x^{-1}$.

18. $216\,a^{-3} + 8\,b^3$ by $6\,a^{-1} + 2\,b$.

19. $625 - 16\,y^4$ by $5 - 2\,y$.

20. $81\,a^4 - 625\,b^{-4}$ by $3\,a - 5\,b^{-1}$.

What combination of signs will give exact division?

21. $a^8 \pm b^8$ by $a^2 \pm b^2$.

22. $a^{12} \pm b^{12}$ by $a^3 \pm b^3$, or $a^4 \pm b^4$.

23. $a \pm b$ by $a^{\frac{1}{3}} \pm b^{\frac{1}{3}}$.

24. $a^2 \pm b^2$ by $a^{\frac{1}{2}} \pm b^{\frac{1}{2}}$.

25. $a^3 \pm b^3$ by $a^{\frac{1}{4}} \pm b^{\frac{1}{4}}$.

Find exact divisors for:

26. $a^{15} + b^{15}$.

27. $32\,x^{-5} + 243\,y^5$.

28. $216\,a^3 + 8\,b^3$.

29. $a^{-3} + b^{-3}$.

30. $27\,a^6 - 8\,b^3$.

TEST EXAMINATIONS.

A.

60. 1. Add $9\,x^4 + 2\,x^3 + x^2 + 12\,x + 1$, $x^4 + 9\,x^3 + 7\,x^2 + x - 9$, $8\,x^4 - 7\,x^3 - x^2 - 15\,x + 10$, and $2\,x^4 + 12\,x^3 + 2\,x^2 + 7\,x + 6$.

2. From the sum of $16\,x^3 - 2\,x - 12$, $-11\,x^3 - 7\,x^2 + 1$, and $9\,x^3 - x^2 - x + 1$ take $2\,x^3 + 2\,x^2 - x + 2$.

3. Reduce to its simplest form $3\,a - [2\,a - 2\,\{a - (a - 1)\} + 2]$.

4. If $a = 2$, $b = 3$, $c = 1$, find the value of $\dfrac{a^2 - b^2}{c} + \dfrac{2\,ab}{\sqrt{3\,b}}$.

5. Multiply $x^6 - a^2x^4 + a^4x^2 - 2\,a^6$ by $2\,x - 3\,a$.

6. Divide $3x^3 - 2x^2y - 5xy^2 + 4y^3$ by $3x + 4y$.

7. Divide $x^6 - y^6$ by $x^2 - y^2$.

8. Find the continued product of $x + y$, $x - y$, $x^2 + xy + y^2$, and $x^2 - xy + y^2$.

B.

1. If $a = 5, b = 3, c = 1$, find the value of $\sqrt{5ab^2} + \sqrt[3]{9bc} - 2\sqrt[4]{3a + b - 2c}$.

2. Square $x^3 - 2x^2 - 3x$ by inspection, and give the rule for the same.

3. Divide $a^6 - 2a^3b^3 + b^6$ by $a^2 - 2ab + b^2$.

4. Reduce to its simplest form $-2a - [3x + \{3c - (4y + 3x + 2a)\}]$.

5. Multiply $x^3 + 2x^2 + 2x + 1$ by $x^3 - 2x^2 + 2x - 1$.

6. Add $9x^2 - 7x + 5$, $-14x^2 + 15x - 6$, and $20x^2 - 40x - 17$.

7. Add $\frac{1}{2}a^3 - 2a^2b - \frac{3}{2}b^3$, $\frac{3}{2}a^2b - \frac{3}{4}ab^2 + 2b^3$, and $-\frac{3}{2}a^3 + ab^2 + \frac{1}{2}b^3$.

8. $(x^{10} - y^{10}) \div (x^2 \pm y^2)$. Use the proper sign in the divisor for exact division, and give the quotient by inspection.

C.

1. From $8a^2 - 2a + 6b^2 - 5ab$ take $a^2 + a + 2b^2 + 2ab$, and find the value of the remainder when $a = 0$, and $b = 3$.

2. From the sum of $-6a^2 + 12b^2 + c^2 + 2bc - 7ac + 2ab$, $a^2 + 5c^2 - 3bc + 8ab$, and $4b^2 - 2c^2 - 3ac - 10ab$ take $5a^2 + 16b^2 + 4c^2 + bc + 10ac$.

3. Divide $64a^{-6} - b^{-6}$ by $2a^{-1} - b^{-1}$.

4. Cube $2x^2 - 3x + 1$ by the formula for cubing any polynomial by inspection.

5. Simplify $3[4x - 5 - 2(3x - 4) + 5\{2x - 3 - (2x - \overline{7x - 5})\}]$.

6. Divide $4x - 10x^{\frac{4}{5}} - 62x^{\frac{3}{5}} - 30x^{\frac{2}{5}}$ by $2x^{\frac{1}{5}} + 5$.

7. The product of two factors is $2x^4 - 13ax^3 + 31a^2x^2 - 38a^3x + 24a^4$; one is $x^2 - 5ax + 6a^2$, find the other.

8. Divide a^4 by $a - 3x$ to 4 terms of quotient.

VII. FACTORING.

CASE I.

61. When all the terms have a common monomial factor.

Factor the following:

1. $x^2 + xy + 2x$. 4. $3x^5 + 9x^3y^2 + 3xy^4$.

2. $2a^3 + ax^2 + ax$. 5. $2ax^2 + 4axy + 2ay^2$.

3. $3x^2y^2 - 2xy - 3xy^2$. 6. $6x^3y^2 + 3x^2y^3 + 3xy^4$.

7. $2x^4 + 6x^3y + 6x^2y^2 + 2xy^3$.

8. $2x^5 + 8x^4y + 12x^3y^2 + 8x^2y^3 + 2xy^4$.

9. $2a^2b^2 + a^3b^3 + ab^4$. 10. $15x^2 + 20xy + 5y^2$.

11. $a^{-2} + a^{-1}b^{-1} + 2a^{-1}b^{-2}$.

12. $x^{-3}y^{-1} - x^{-2}y^{-2} + x^{-1}y^{-3}$.

13. $2a^4b^{\frac{3}{4}} - 3a^3b^{\frac{1}{2}} + a^2b^{\frac{1}{2}}$.

14. $a^{\frac{3}{2}} + ab + a^{\frac{1}{2}}b^{\frac{1}{2}}$.

15. $x^{-4}y^{-4} + x^{-2}y^{-2} + x^{-1}y^{-1}$.

16. $3a^4b^{-4} + 6a^2b^{-2} + 3ab^{-1}$.

17. $20a^4b^4c^2 + 30a^3b^3c^2 + 20a^2b^2c^2$.

18. $27a^{-3}b^{-3} + 54a^{-2}b^{-2} + 81a^{-2}b^{-1}$.

19. $16a^2b^2 + 48a^2b - 16ab + 8a$.

20. $a^{-7}b + 4a^{-5}b^{-1} + 6a^{-3}b^{-3} + 4a^{-1}b^{-5} + ab^{-7}$.

CASE II.

62. Trinomials which can be factored by finding what two factors multiplied together give the third term, while the sum of the same two gives the coefficient of the second term. (The converse of § 47.)

A.

Factor the following:

1. $x^2 + 12x + 35.$

2. $x^2 + 11x + 24.$

3. $a^2 + 15a + 56.$

4. $m^2 + 16m + 63.$

5. $x^2 + 30x + 225.$

6. $x^4 + 8x^2 + 7.$

7. $x^{10} + 20x^5 + 19.$

8. $x^{-2} + 14x^{-1} + 45.$

9. $a^{-4} + 15a^{-2} + 36.$

10. $a^{-1} + 5a^{-\frac{1}{2}} + 6.$

11. $x^{-\frac{2}{3}} + 10x^{-\frac{1}{3}} + 21.$

12. $m^{-6} + 2m^{-3} + 1.$

13. $a^4x^4 + 18a^2x^2 + 17.$

14. $x^{\frac{1}{2}} + 6x^{\frac{1}{4}} + 5.$

B.

Factor the following:

15. $x^2 - 13x + 40.$

16. $x^2 - 7x + 6.$

17. $x^2 - 9x + 20.$

18. $x^8 - 19x^4 + 88.$

19. $x^{-6} - 10x^{-3} + 21.$

20. $x^{-2} - 24x^{-1} + 143.$

21. $a^{-\frac{2}{3}} - 12a^{-\frac{1}{3}} + 20.$

22. $a^4x^{-2} - 3a^2x^{-1} + 2.$

23. $x^4 - 21x^2 + 90.$

24. $x^{-8} - 33x^{-4} + 260.$

25. $x^{10} - \frac{7}{6}x^5 + \frac{1}{3}.$

26. $x - 15x^{\frac{1}{2}} + 50.$

C.

Factor the following:

27. $x^2 + 3x - 40.$

28. $a^2 + 8a - 20.$

29. $x^{-4} + 3x^{-2} - 10.$

30. $x^{-2} + 3x^{-1} - 40.$

31. $x^2 + 3x - 180.$

32. $x^2 + 8x - 65.$

33. $x^2 + x - 2.$

34. $x^2 + \frac{1}{6}x - \frac{1}{6}.$

35. $x^6 + x^3 - 42.$

36. $x^4y^2 + 6x^2y - 40.$

37. $x^{-4}y^{-2} + 3x^{-2}y^{-1} - 28.$

38. $x^{-\frac{1}{2}} + 2x^{-\frac{1}{4}} - 8.$

D.

Factor the following:

39. $x^2 - 3x - 108.$

40. $x^2 - 2x - 323.$

41. $x^4 - 7x^2 - 8$

42. $x^{-2} - 2x^{-1} - 99.$

43. $a^4 - 4a^2 - 165.$

44. $a^2b^2 - 3ab - 40.$

45. $a^4b^2 - 2a^2b - 63.$

46. $a^{-4} - 13a^{-2} - 14.$

E.

Perfect Squares. Special form of (A).
(The converse of § 49.)

Factor the following:

47. $x^2 + 10x + 25.$

48. $x^2 + 20x + 100.$

49. $x^{10} + 2x^5 + 1.$

50. $x^2y^2z^2 + 8xyz + 16.$

51. $x^8 + 12x^4 + 36.$

52. $a^{12} + 4a^6 + 4.$

53. $a^2 + 16a + 64.$

54. $x^{10} + 30x^5 + 225.$

55. $x^{16} + 40x^8 + 400.$

56. $x^{-2} + 2x^{-1} + 1.$

57. $x^{-4} + 4x^{-2} + 4.$

58. $x + 6x^{\frac{1}{2}} + 9.$

59. $x^{\frac{2}{5}} + 10x^{\frac{1}{5}} + 25.$

60. $x^{-6} + 18x^{-3} + 81.$

61. $x^{-1} + 36x^{-\frac{1}{2}} + 324.$

62. $a^4x^{-2} + 8a^2x^{-1} + 16.$

63. $a^{2m} + 12a^m + 36.$

64. $a^{-2}x^{-m} + 14a^{-1}x^{-\frac{m}{2}} + 49.$

F.

Perfect Squares. Special form of (B).

(The converse of § 50.)

Factor the following:

65. $x^2 - 28x + 196$.

66. $x^2 - 30x + 225$.

67. $x^2 - 60x + 900$.

68. $x^8 - 18x^4 + 81$.

69. $x^6 - 24x^3 + 144$.

70. $x^{10} - 28x^5 + 196$.

71. $x^{-4} - 50x^{-2} + 625$.

72. $x^{-2}y^{-2} - 80x^{-1}y^{-1} + 1600$.

73. $x^{-6} - 56x^{-3} + 784$.

Case III.

63. Trinomials, the coefficient of whose highest power is not unity.

A.

Factor the following:

1. $2x^2 - 3x - 2$.

2. $2x^2 + 5x + 2$.

3. $3x^2 + 2x - 1$.

4. $4x^2 - 11x + 6$.

5. $5x^2 - 11x + 2$.

6. $56x^2 + 9x - 2$.

7. $56x^2 + 26x - 1$.

8. $16x^2 + 2x - 3$.

9. $56x^2 - 9x - 2$.

10. $35x^2 + 17x + 2$.

11. $6x^4 - 13x^2 + 6$.

12. $8x^6 + 2x^3 - 1$.

13. $15x^2 + 24x - 12$.

14. $55x^4 - x^2 - 2$.

15. $12x^2 - 25x + 12$.

16. $24x^2 + 18x + 3$.

17. $6x^2 + 7xy + 2y^2$.

18. $21x^2 + xy - 2y^2$.

19. $4x^4 - 2x^2y^2 - 6y^4$.

20. $6x^2y^2 - 5xyz - 6z^2$.

B.

Perfect squares. Special form of (A).

(The converse of § 49.)

Factor the following:

1. $4x^2 + 12x + 9$.
2. $9x^2 + 42x + 49$.
3. $25x^2 + 10x + 1$.
4. $9x^2 + 48x + 64$.
5. $49x^2 + 28x + 4$.
6. $9x^2 + 6x + 1$.
7. $25x^2 + 20x + 4$.
8. $49x^2 + 42x + 9$.
9. $64x^2 + 32x + 4$.
10. $9x^2 + 12xy + 4y^2$.
11. $9x^4 + 18x^2y^2 + 9y^4$.
12. $64x^4 + 16x^2y^2 + y^4$.

C.

Perfect Squares. Special form of (A).

(The converse of § 50.)

Factor the following:

1. $4x^2 - 4xy + y^2$.
2. $9x^2y^2 - 12xy^2z + 4y^2z^2$.
3. $64x^4y^2 - 48x^3y^3 + 9x^2y^4$.
4. $16x^4y^4z^4 - 8x^3y^3z^3 + x^2y^2z^2$.
5. $36x^{-2} - 24x^{-1}y^{-1} + 4y^{-2}$.
6. $100x - 40x^{\frac{1}{2}}y^{\frac{1}{2}} + 4y$.
7. $81x^{\frac{4}{3}} - 18x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$.
8. $9x^{-8} - 30x^{-4}y^{-4} + 25y^{-8}$.
9. $9x^2 - 6xy^{\frac{1}{2}} + y$.
10. $9x^{-2}y^{-2} - 30 + 25x^2y^2$.
11. $4x^{-4}y^{-4} - 16x^{-1}y^{-1} + 16x^2y^2$.
12. $16a^2x - 48a^{\frac{3}{2}}x^{\frac{3}{2}} + 36ax^2$
13. $4x^2 - 12xy^{-1} + 9y^{-2}$.
14. $81a^{-4} - 144a^{-2}b^{-1} + 64b^{-2}$.

CASE IV.

64. The sum or difference of like powers. (See § 59.)

Factor the following:

1. $x^3 + y^3$.
2. $x^3 - y^3$.
3. $x^{10} + y^{10}$.
4. $x^{15} + y^{15}$.
5. $8x^3 + y^3$.
6. $x^3 - 343y^6$.
7. $32x^5 + 243y^5$.
8. $128x^7 + y^7$.
9. $a^6 + 64b^6$.
10. $125a^3 + 27b^3$.
11. $27x^6 - 8y^6$.
12. $a^{-9} + b^{-9}$.
13. $8a^{-6} + 27b^{-6}$.
14. $1728a^{-3} + 64b^{-3}$.
15. $243x^{10} + y^{15}$.

CASE V.

65. The difference of two squares.

(The converse of § 51.)

Factor the following:

1. $x^2 - y^2$.
2. $x^6 - y^6$.
3. $x^4 - y^4$.
4. $16x^2 - 25y^2$.
5. $81y^4 - 64z^4$.
6. $(a + b)^2 - (a - b)^2$.
7. $a^2 - 2ab + b^2 - c^2$.
8. $a^4 - 2a^2b^2 + b^4 - 1$.
9. $1 - a^2 - 2ab - b^2$.
10. $(a - b)^2 - c^2$.
11. $(2x + 3y)^2 - (2x - 3y)^2$.
12. $a^2 + b^2 - b^2 - c^2 + 2ab + 2bc$.
13. $a^2 - d^2 + c^2 - e^2 - 2ac + 2de$.
14. $a^2 + 2a + 2be - b^2 - e^2 + 1$.
15. $a^2 + b^2 - m^2 - n^2 - 2ab - 2mn$.
16. $x^2 - (x + y)^2$.
17. $(3x - 2)^2 - (2x - 3)^2$.
18. $2xy - x^2 - y^2 + 1$.
19. $x^2 - 2yz - y^2 - z^2$.
20. $a^2 - x^2 + 2x - 1$.
21. $9x^2 + 6xy - 4z^2 + y^2$.
22. $a^{16} - b^{16}$ into five factors.
23. $a^2 - b^2 - c^2 + d^2 - 2(ad - bc)$.

24. $16x^{-4} - 16y^{-4}$. **25.** $x^{-\frac{2}{3}} - y^{-\frac{2}{3}}$.

26. $a^{-2} - 2a^{-1}b^{-1} + b^{-2} - c^{-2}$.

27. $(a^{-2} - b^{-2})^2 - b^{-4}$.

28. $9a^2 - 12ab + 4b^2 - 16x^2 - 8xy - y^2$.

29. $a^2 - 2ab + b^2 + 2cd - c^2 - d^2$.

30. $2cd - c^2 - d^2 + a^2 + b^2 + 2ab$.

31. $4x^2y^2 - (x^2 + y^2 - z^2)^2$.

32. $4(ad + bc)^2 - (a^2 - b^2 - c^2 + d^2)^2$.

33. $(a^2 + 9b^2 - c^2)^2 - 36a^2b^2$.

34. $64x^2y^2 - (x^2 + 16y^2 - z^2)^2$.

CASE VI.

66. Expressions which can be arranged in groups of two or more terms so that all the groups shall contain a common factor.

Factor the following:

1. $a^2 - ax + ab - bx$. **5.** $6x^2 - 9x - 10xy + 15y$.

2. $ax + ay - bx - by$. **6.** $2ax - 3bx + 2a - 3b$.

3. $x^2 - 2xy + xz - 2yz$. **7.** $xy + x + y + 1$.

4. $2x^2 - 2xy - x + y$. **8.** $x^2y + 2xy - 2x^2 - 4x$.

9. $a^4 + a^2y^2 - a^2b^2 - b^2y^2$.

10. $8x^2 - 12ax - 6xy + 9ay$.

11. $x^4 - 2x^3 + 7x - 14$. **13.** $x^5 + 5x^4 + x^2 + 5x$.

12. $x^3 + 3x^2 - 5x - 15$. **14.** $2x^3 + x^2 + 6x + 3$.

15. $14x^3 - 6x^2 - 21x + 9$.

16. $2a^2x^2 + 3a^2x + 2b^2x^2 + 3b^2x + a^2 + b^2$.

17. $3ax - 3ab - 2x^2 + 2bx - b + x$.

18. $x^2(a - b) + x(b - a) + 2(a - b)$.

19. $2x - ax - ay - az + 2y + bx + 2z + by + bz$.

20. $x^3 + (a + 1)x^2 + (a + 1)x + a$.

CASE VII.

67. Trinomials in the form of $a^{2n} \pm a^nb^n + b^{2n}$, to be reduced to the difference of two squares, and factored by Case V.

Factor the following:

1. $x^4 - 6x^2y^2 + y^4$.
2. $x^4 + x^2 + 1$.
3. $a^4 - 7a^2b^2 + b^4$.
4. $16x^4 - 17x^2y^2 + y^4$.
5. $9a^4 + 21a^2b^2 + 25b^4$.
6. $49x^4 - 15x^2y^2 + 121y^4$.
7. $25x^4 + 66x^2y^2 + 49y^4$.
8. $121x^8 - 180x^4y^4 + 64y^8$
9. $36a^4 - 28a^2b^2 + 4b^4$.
10. $64a^4 + 60a^2b^2 + 16b^4$.
11. $16x^{-4} + 36x^{-2}y^{-2} + 25y^{-4}$.
12. $4a - 16a^{\frac{1}{2}}b^{\frac{1}{2}} + 9b$.
13. $25a^{-4} - 94a^{-2}b^{-2} + 81b^{-4}$.
14. $100a^{-2} + 176a^{-1}b^{-1} + 81b^{-2}$.
15. $a^6 - 14a^3y^3 + y^6$.
16. $36a^2 - 88ab + 49b^2$.
17. $x^4 - 3x^2 + 1$.
18. $4x^4 + 3x^2y^2 + 9y^4$.

CASE VIII.

68. Expressions with two trinomial factors.

Factor the following:

1. $2x^2 + 5xy + 2y^2 - 8x - 7y + 6$.
2. $6x^2 - 5xy - 6y^2 + 4x + 7y - 2$.
3. $15x^2 + 9x + 13xy - 6y^2 + 20y - 6$.
4. $14x^2 + 3x + 4y^2 - 2y + 18xy - 2$.
5. $6x^2 + x - 2y^2 - 4y + xy - 2$.
6. $13xz + yz - xy + 6x^2 + 6z^2 - y^2$.
7. $10x^2 - 3x + 9xy - y + 2y^2 - 1$.
8. $16xz + 15yz + 9y^2 + 6z^2 + 18xy + 8x^2$.
9. $8x^2 - 9xy - 9xz + 2yz + y^2 + z^2$.
10. $4x^2 + 4xy - 2xz - 3y^2 - 7yz - 2z^2$.
11. $15x^2 - 16xy - 2x + 4y^2 - 1$.
12. $2x^4 + 3x^2y^2 + 5x^2z^2 + 3y^2z^2 + y^4 + 2z^4$.

MENTAL EXERCISES IN FACTORING.

Factor the following:

69. 1. $a^2 - b^2$.
2. $36\,a^2 - 49b^2$.
3. $a^2 + 11a + 30$.
4. $a^2 - a - 6$.
5. $a^3 - b^3$.
6. $a^3 + b^3$.
7. $a^4 - b^4$.
8. $a^5 - b^5$.
9. $(a + b)^2 - (a - b)^2$.
10. $a^2 + 16\,a + 64$.
11. $x^2 - 8\,x + 16$.
12. $x^3 + 3\,x^2 + 3\,x + 1$.
13. $49x^2 + 28\,xy + 4\,y^2$.
14. $4\,x^2 - 12\,xy + 9\,y^2$.
15. $x^2 - a^2 - 2\,ab - b^2$.
16. $a^2 - 2\,ab + b^2 - x^2$.
17. $a^2 - 2\,ab + b^2 - x^2 + 2\,xy - y^2$.
18. $x^2 + y^2 - a^2 - b^2 - 2\,xy - 2\,ab$.
19. $(x + y)^2 - (a - b)^2$.
20. $2\,abx + 3\,aby + 2\,a^2b^2x + 3\,a^2b^2y$.
21. $a^3 + 3\,a^2b + 3\,ab^2 + b^3$.
22. $x^3 - 27$.
23. $a^2 + ac + 3\,ab + 3bc$.
24. $2\,ax - 6\,x - 2\,ay + 6\,y$.
25. $a^4 + 4a^3b + 6a^2b^2 + 4\,ab^3 + b^4$.
26. $x^3 + 27$.
27. $x^2 + x - 6$.
28. $x^2 - 5\,x + 6$.
29. $4\,x^2 + 12\,xy + 9y^2$.
30. $4\,x^2 + 4\,xy + y^2$.
31. $a^2 - b^2 + a - b$.
32. $a^3 - 8\,b^3$.
33. $8\,a^3 - b^3$.
34. $a^4 - b^8$.
35. $ax - ay + bx - by$.
36. $(a + b)^2 + 2\,(a + b) + 1$.
37. $a^5 - ab^4$.
38. $ax + ay + bx + by$.
39. $12\,a^2b - 75\,b^5$.
40. $x^2 + 2x + x + 2$.
41. $ax + bx + ac + bc$.
42. $x^2 - y^2 + (x - y)^2$.
43. $x^4 - y^2$.
44. $x^4 - 6\,x^2y^2 + y^4$.
45. $x^4 + x^2y^2 + y^4$.

GENERAL EXERCISES IN FACTORING.

Factor the following:

70. **1.** $6x^3 - 2x^2 - 21x + 7$.

2. $8x^3 - 6x^2 - 28x + 21$.

3. $(x^2 - 2x - 1)^2 - (x^2 - 1)^2$.

4. $x^3 - 2x^2 + 3x - 6$.

5. $(a^2 + b^2 - c^2)^2 - 4a^2b^2$.

6. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.

7. $(2x + y - z)^2 - (x - y + z)^2$.

8. $x^2 - 2x + 1 - y^2 - 4yz - 4z^2$.

9. $x^4y + x^2y^3 - x^3y^2 - xy^4$.

10. $(x + y)^3 - (x - y)^3$.

11. $x^2 - y^2 + (x - y)^2$.

12. $(a + 3b)^4 - (a - 3b)^4$.

13. $x^4 - 167x^2y^2 + y^4$.

14. $x^8 - 7x^4y^4 + y^8$.

15. $ax^4 - ax^2y^2 + ax^3y - axy^3$.

16. $(a^4 - 2a^2b^2 - b^4)^2 - 4a^4b^4$.

17. $a^2 - b^2 - c^2 + d^2 - 2(ad - bc)$.

18. $(x + y)^2(x - y) - (x - y)^2(x + y)$.

19. $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$.

20. $x^2 - 4y^2 + 12yz - 9z^2$.

21. $x^2 - 6xy + 9y^2 - 2xz + 6yz + z^2$.

22. $x^4 + 2x^2z - 4y^2 + z^2$.

23. $8x^3 - 12x^2 + 6x - 1$.

24. $6x^2 - 7xy + 2y^2$.

25. $20x^3 + 26x^2y - 6xy^2$.

26. $343x^3 - 1000y^3$.

27. $64x^6 - 729y^6$.

28. $(x + y)^4 - 1$.

29. $m^4 - (m + n)^4$.

30. $x^2 + 4ax + 4a^2 - b^2$.

31. $x^2 + 5ax + 6a^2 + a - 1.$

32. $x^2 - (2a - 1)x + a^2 - a - 2.$

33. $x^2 + 5ax + 6a^2 + ab - b^2.$

34. $(x^2 - 11x + 30)^2 - (x^2 - x - 30)^2.$

35. $4a^3 - a^2 - 8a + 2.$

36. $x^4 - 27x^2y^2 + y^4.$

37. $32x^5 - 1.$

38. $32x^5 - 243y^5.$

39. $x^{16} - y^{16}.$

40. $x^{18} - y^{18}.$

41. $a^{30} - b^{30}.$

42. $\dfrac{x^2}{a^2} + \dfrac{a^2}{x^2} - 2.$

43. $x^2 + 4y^2 - 9z^2 - 4xy.$

44. $x^3 + x^2y + x^2z - xyz - y^2z - yz^2.$

45. $2x^5 + 2x^4 + 3x^3 + 3x^2 + 4x + 4.$

46. $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5.$

47. $a^2x + abx + ac + aby + b^2y + bc.$

48. $a^2b + a^2x + acy + ad + abx + ax^2 + cxy + dx.$

49. $16^x + 2 \cdot 2^{2x} + 1.$

50. $2^{5x} - 32.$

51. $x^2 + y^2 + z^2 - u^2 + 2xy + 2yz + 2xz.$

52. $a^2 + b^2 + c^2 - d^2 - 2ab - 2ac + 2bc.$

53. $a^2 + b^2 + c^2 - x^2 - y^2 - 2ab - 2bc + 2ac + 2xy.$

54. $acx^2 + (ad + bc)x + bd.$

55. $1 - x - x^2 + x^3.$

56. $1 - x^8 - x^{16} + x^{24}.$

57. $a^{16} + a^8x^8 + x^{16}.$

58. $5x^3 + 2x^2 - 15x - 6.$

59. $8(x + y)^3 - (2x - y)^3.$

60. $4(x - y)^3 - x + y.$

61. $(x + y)(x - y)^3 - x^4 + y^4.$

62. $(x + y)(x - y)^5 - x^6 + y^6.$

63. $a(a - 1)^2 - a^2(a - 1) + a(a - 1).$

64. $4^{4x} + 8^{3x} + 32^{2x} + 2^{11x} + 64^{2x}$.

65. $x^2 - 3bcx - acx - abx + 3ab^2c + 3abc^2$.

66. $x^2 + 2a^2x + bcx + acx + 2a^3c + 2a^2bc$.

67. $(x^2 - 5x)^2 + 6x^2 - 30x$.

68. $x^2 - (y + 2)x - 2(y^2 + 7y + 12)$.

69. $x^2 + cx(a + b) - ab(a - c)(b + c)$.

70. $x^2 - (2a - 1)x - 6 - a + a^2$.

71. $x^2 - ax - 2bx + ab + b^2$.

72. $x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4$.

73. $36a^2 - 24ab + 4b^2 - 12bc + 36ac + 9c^2$.

74. $x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$.

75. $x^4 - 11x^2 + 1$.

76. $(x - 2y)x^3 - (y - 2x)y^3$.

77. $x^4 + 16a^2x^2 - 8ax^3 - 16a^4$.

78. $x^3y^3 - 512$. **80.** $(x + y)^5 - x^5 - y^5$.

79. $x^4y - x^2y^3 - x^3y^2 + xy^4$. **81.** $(x + y)^7 - x^7 - y^7$.

TEST EXAMINATIONS IN FACTORING.

A.

Factor the following:

71. **1.** $12x^2 - 11xy + 2y^2$.

2. $32x^3 - 48x^2y + 18xy^2$.

3. $x^2 - a^2 - b^2 + y^2 - 2xy - 2ab$.

4. $ax^5 - bx^5 - ay^5 + by^5$. **7.** $64x^2y^2 - (x^2 + 16y^2 - z^2)^2$.

5. $x^4 - 11x^2y^2 + y^4$. **8.** $2a^2 - 2ab + ax - bx$.

6. $x^{15} + y^{15}$ (give 3 sets). **9.** $64a^6 - 729b^6$.

10. $a^2 + 2ab + b^2 + 3a + 3b - 10$.

B.

1. $x^{12} - y^{12}, x^{12} + y^{12}$.

2. $6x^2 - 13xy + 6y^2 + 12x - 13y + 6$.

3. $3x^2 + xy - 2y^2$. 5. $x^8 - 47x^4y^4 + y^8$.

4. $x^5 + y^5 - x^4y - xy^4$. 6. $x^3 + y^3 + x^2 - xy + y^2$.

7. $256x^4y^4 - (x^4 + 64y^4 - z^4)^2$.

8. $a^3 - b^3 + a^4 + a^2b^2 + b^4$.

9. $x^3 + y^3 + x^2y + xy^2 + x + y$.

10. $x^2 + y^2 - z^2 - u^2 - 2zu + 2xy$.

VIII. HIGHEST COMMON FACTOR.

DEFINITIONS.

72. A **Common Factor** of two or more algebraic expressions is an expression that will exactly divide each of them.

73. The **Highest Common Factor** of two or more algebraic expressions is the expression of *highest degree* that will exactly divide each of them.

QUESTIONS.

74. 1. What factors are found in the H. C. F.?

2. How is the H. C. F. found when the expressions can be factored by inspection?

3. How, when they cannot be factored by inspection? Give the method for two expressions; for more than two.

MENTAL EXERCISES.

Find the H. C. F. of the following:

75. 1. $a^2 + 2ab + b^2,\ a^2 - b^2,\ a^3 + b^3,\ a^4 - b^4.$

2. $12\,(x-1)^2,\ 6\,(x-1)^3,\ 4\,(x^2 + x - 2),\ 2\,(x^2 - 1).$

3. $a^2 + 4\,a + 4,\ a^2 + 3\,a + 2,\ a^2 + a - 2.$

4. $a^3 + 3\,a^2 + 3\,a + 1,\ a^4 + 4\,a^3 + 6\,a^2 + 4\,a + 1.$

5. $x^4 + 2a^2x^2 + a^4,\ x^4 - a^4,\ 2\,(x^{10} + a^{10}).$

6. $x^3 + y^3,\ x^2 - xy + y^2,\ x^6 - y^6.$

7. $a^2 + 3\,a - 4,\ a^2 - 7\,a + 6,\ a^2 - 2\,a + 1.$

8. $15\,(x+y)^2,\ 12\,(x^2 - y^2)^2,\ 9\,(x^2 - y^2).$

GENERAL EXERCISES.

CASE I.

76. Expressions which can be factored by inspection.

Find the H. C. F. of the following:

1. $a^2 - b^2$, $a^3 - b^3 - 3ab(a - b)$.
2. $a^2 + ab - 2b^2$, $a^2 + 6ab + 8b^2$.
3. $a^2 + 3ab + 2b^2$, $a^2 - ab - 2b^2$, $a^5 + b^5$.
4. $ab - ac + 2b - 2c$, $a^3 + 6a^2 + 12a + 8$.
5. $a^2 - 2a + ab - 2b$, $3a^2 - 5a - 2$.
6. $a^4 + a^2b^2 + b^4$, $a^2x + a^2y + abx + aby + b^2x + b^2y$.
7. $6x^2 - 5x - 6$, $2ax - 2bx - 3a + 3b$.
8. $2ax + 2bx - 3ay - 3by$, $ax + bx - 2a - 2b$.
9. $x^4 - y^4$, $x^5 + y^5$, $x^7 + y^7$, $x^2 + 2xy + y^2$.
10. $a^2 - ab - 6b^2$, $a^2 - 2ab - 8b^2$, $a^2 - 3ab - 10b^2$.
11. $2ax - 10a - bx + 5b$, $3ax - 15a - 2bx + 10b$.
12. $a^3 + 3a^2b + 3ab^2 + b^3$, $2a^2 + ab - b^2$.
13. $12x^2 - 15xy + 3y^2$, $6x^3 - 6x^2y + 2xy^2 - 2y^3$.
14. $y^3 - 2y^2 + 3y - 6$, $y^4 - 2y^3 - y^2 + 2y$.
15. $x^3 - 2x^2 - 8x + 16$, $x^3 + 3x^2 - 8x - 24$.
16. $x^3 + 3x^2 - 7x - 21$, $x^3 + 3x^2 - 3x - 9$.
17. $x^5 - x^3 - x + 1$, $ax^4 + 2bx^4 + ax^3 - a + 2bx^3 - 2b$.
18. $4ax - 6x - 2ay + 3y$, $4a^2 - 10a + 6$.
19. $8a^3 + 36a^2x + 54ax^2 + 27x^3$, $4a^2 + 8ax + 3x^2$.
20. $x^6 + x^4y^2 + x^2y^4 + y^6$, $x^3 + 2x^2y + xy^2 + 2y^3$.
21. $x^{15} - y^{15}$, $x^9 - y^9$, $x^6 - y^6$, $x^3 - y^3$.
22. $a^2x - a^2y + 2abx - 2aby + b^2x - b^2y$,
    $a^2x - a^2y - b^2x + b^2y$.

Case II.

77. Expressions, one or more of which can be factored by inspection.

Find the H. C. F. of the following:

1. $11a^2 + 87ay - 8y^2$, $22a^3 - 13a^2y - 10ay^2 + y^3$.
2. $2a^2 - 13a - 7$, $2a^3 - a^2 + 3a + 2$.
3. $7x^3 - 25x^2 - 23x + 20$,
   $ax^2 + bx^2 - 3ax - 3bx - 5a - 5b$.
4. $14x^3 - 19x^2 - 10x - 1$,
   $4ax^2 - 2bx^2 - 6ax + 3bx - 2a + b$.
5. $8x^4 - 6x^3 - 28x^2 + 21x$, $6x^5 - 25x^3 + 14x$.
6. $25x^6 + 40x^4 + 16x^2$, $10x^8 - 7x^6 - 2x^4 + 8x^2$.
7. $27x^3 + 6x - 3$, $27x^2 - 3$, $27x^2 - 18x + 3$.
8. $4a^3 + 8a^2 + 8a + 4$, $8a^2 + 12a + 4$.
9. $a^5 - 2a^4 - a^3 - 2a^2 + a$, $a^5 - a^4 - a^2 + a$.
10. $x^5 - x^4y - xy^4 + y^5$, $x^4 - x^3y - x^2y^2 + xy^3$,
    $x^3 - x^2y - xy^2 + y^3$.
11. $x^3 + 4x^2 - 47x - 210$, $x^2 - x - 42$.
12. $a^3 + 9a^2 + 26a + 18$, $a^3 + a^2 + a + 1$.
13. $a^2 + ac + ab + bc$, $2a^2 - ab - 3b^2$.
14. $a^4 + 10a^3b + 35a^2b^2 + 50ab^3 + 24b^4$, $a^2 - 2ab - 3b^2$.
15. $2x^4 + 3x^3 - 2x^2 + 12x + 5$,
    $ax^2 + bx^2 + 3ax + 3bx + a + b$.
16. $2x^2 - 15x + 25$, $x^3 - 5x^2 - x + 5$.
17. $2a^2x^2 + a^2x - a^2$, $ax^4 + ax^3 + ax^2 + ax$.
18. $2x^4 - 6x^3 + 3x^2 - 3x + 1$, $a^2x^2 - 3a^2x + a^2$.
19. $2x^4 - 3x^3 + 7x^2 - 12x - 4$,
    $2mx^2 - 3mx + 2nx^2 - 3nx - m - n$.
20. $a^2x - b^2x$, $a^2 - 3ab + 2b^2$, $a^3 - a^2b - ab + b^2$.

CASE III.

78. Expressions not readily factored by inspection.

Find the H. C. F. of the following:

1. $2a^4 - 6a^3 + 3a^2 - 3a + 1$, $a^7 - 3a^6 + a^5 - 4a^2 + 12a - 4$.

2. $x^4 + x^3 - 53x + 51$, $x^4 - 12x^2 + 8x + 3$.

3. $x^3 + 10x^2 + 33x + 36$, $x^3 + 9x^2 + 23x + 15$.

4. $5x^3 - 21x^2 + 5x - 4$, $5x^3 - 19x^2 - 5x + 4$.

5. $x^3 - 13x^2 + 55x - 75$, $2x^3 - 19x^2 + 55x - 50$.

6. $x^3 - 2x^2 - 41x + 42$, $x^4 + 6x^3 - 39x^2 - 388x + 420$.

7. $2x^4 + 3x^3 - 3x^2 - 5x - 6$, $2x^3 + 3x^2 + 3x + 1$.

8. $9x^4 + 9x^3 + 2x^2 + x - 1$, $9x^3 - 15x^2 + 7x - 1$.

9. $a^6 - 2a^5 - 4a^3 - 2a + 1$, $a^5 + a^4 - 2a^3 + a^2 + a + 4$.

10. $x^3 - 10x^2 - 25x + 250$, $2x^3 - 3x^2 - 65x + 150$.

11. $9x^4 + 51x^3 + 45x^2 - 119x - 154$,
$3x^3 + 8x^2 - 8x + 11$.

12. $12x^4 - 22x^3 - 15x^2 + 25x - 6$,
$24x^4 - 14x^3 - 34x^2 + 30x - 6$.

13. $5x^7 - 5x^6 - x^5 + x^4 + 11x^3 - 11x^2 + 12x - 12$,
$10x^6 - 7x^4 - 2x^2 + 8$.

14. $12x^4 - 40x^3 - 25x^2 + 85x - 12$,
$9x^3 - 12x^2 - 15x + 20$, $6x^3 - 29x^2 + 31x - 4$.

IX. LOWEST COMMON MULTIPLE.

DEFINITIONS.

79. A **Multiple** of an algebraic expression is any expression exactly divisible by the given expression.

80. A **Common Multiple** of two or more algebraic expressions is an expression exactly divisible by each of them.

81. The **Lowest Common Multiple** of two or more algebraic expressions is the expression of *lowest degree* exactly divisible by each of them.

QUESTIONS.

82. 1. What factors are found in the L. C. M.?

2. How is the L. C. M. found when the expressions can be factored by inspection?

3. How, when they cannot be factored by inspection? Give the method for two expressions; for more than two.

MENTAL EXERCISES.

Find the L. C. M. of the following:

83. 1. $x - y, x^2 + xy + y^2, x^3 - y^3$.

2. $x + y, x^2 - xy + y^2, x^3 + y^3$.

3. $y - 3, y^2 + 3y + 9, y^5 - 27$.

4. $3x + 1, 9x^2 + 6x + 1, 27x^3 + 27x^2 + 9x + 1$.

5. $9x^2 - 12x + 4, 9x^2 - 4, 3x^2 + 2x$.

6. $4a^2 - 12ax + 9x^2, 4a^2 - 9x^2, 2a - 3x$.

7. $12a^3b^2c^3, 16a^4bcd^2, 15a^2b^2c^2, 6abcd$.

8. $20a^5bcx^2y^2, 12a^3b^3c^3d^3, 15abc^2dxy$.

GENERAL EXERCISES.

Find the L. C. M. of the following:

84. 1. $x^2 - 2x + 1, x^2 - 3x + 2, x^2 + x - 2$.

2. $2(x + 1)^2, 4(x^3 - 1), 3(x^2 - 1), 2(x^2 + x - 2)$.

3. $2x^2 - x - 1,\ 2x^2 + 5x + 2,\ 2x^2 + 7x + 3.$

4. $x^2 - 7x + 10,\ x^2 - 8x + 15, 4x^3 - 25x^2 + 20x + 25.$

5. $x^2 + 4x - 21,\ x^2 - 3x - 70,\ x^3 - 39x + 70.$

6. $x^3 - y^3,\ x^3 + y^3,\ x^6 - y^6,\ x^{12} - y^{12}.$

7. $x^2 - 4,\ x^2 - 7x + 10,\ x^3 - 5x^2 + 4x - 20.$

8. $a^4 - a^3 + 2a^2 - 3a - 3,\ 2a^3 - 3a^2 - a + 1.$

9. $x^3 - x^2 + x + 3,\ x^3 - x + 6.$

10. $x^2 + 5x + 4,\ x^2 + 4x + 3,\ x^5 + 5x^3 + 6.$

11. $x^2 + 2x - 3,\ 3x^2 - 5x + 2,\ 4x^3 - 4x^2 - x + 1.$

12. $x^3 + 2x^2 - 9x - 18,\ x^2 + x - 2,\ x^2 - 4x + 3.$

13. $4x^3 + 8x^2 - x - 2,\ 2x^2 + 15x - 8,\ x^2 + 10x + 16.$

14. $12x^2 + 3x - 42,\ 12x^3 + 30x^2 + 12x,$
$32x^2 - 40x - 28.$

15. $2x^2 - x - 10,\ 2x^2 + x - 15,\ 4x^2 - 20x + 25.$

16. $3x^3 + 14x^2 - 19x - 70,\ 3x^3 + 2x^2 - 51x + 70.$

17. $5(x^2 - 1), 8(x^3 - 1), x^2 - 2x + 1, x^3 - 3x^2 + 3x - 1.$

18. $2ax^2 - 2ax - 12a,\ 3x^2 + 3x - 6,\ 5x^2 - 20x + 15.$

19. $15x^2 + 55x + 30,\ 12x^3 + 32x^2 + 16x,\ x^2 + 5x + 6.$

20. $9ax^3 - 3ax^2 - 42ax,\ 6x^2 - 26x + 28,\ x^2 - 4.$

21. $16x^2 - 76xy + 70y^2,\ 12x^2 - 3xy - 15y^2,$
$8x^2 - 20xy - 28y^2.$

22. $8x^2 + 12x - 8,\ 6x^2 + 45x - 24,$
$x^3 + 15x^2 + 66x + 80.$

23. $24ax^2 - 46axy + 20ay^2,\ 12bx^2 - 27bxy + 15by^2,$
$6cx^2 - 10cxy + 4cy^2.$

24. $x^3 + 2x^2 - 5x - 6,\ x^3 - 3x^2 + 4,$
$x^4 - 2x^3 - 3x^2 + 4x + 4.$

X. FRACTIONS.

85. Fractions in Algebra are treated in precisely the same way as in Arithmetic. The definitions and rules given by different authors vary somewhat, and are an attempt to adapt the corresponding arithmetical forms to the algebraic notation. For all ordinary purposes the arithmetical definitions and rules may be used without change by remembering that instead of numbers we are dealing with algebraic expressions, and that all the operations are algebraic.

86. REDUCTION TO LOWEST TERMS.

Reduce the following fractions to their lowest terms:

1. $\dfrac{6x^2+x-2}{6x^2-x-1}$. 2. $\dfrac{6x^2-54}{6x^2+36x+54}$.

3.
$$\frac{x^2-a^2-b^2+y^2-2xy-2ab}{x^2+y^2+a^2+b^2-2xy-2ax-2bx+2ay+2by+2ab}.$$

4. $\dfrac{1-a^2-2ab-b^2}{1+a^2+b^2+2a+2b+2ab}$.

5. $\dfrac{(x-y)^2-(c-d)^2}{x^2-2xy+y^2+cx-cy-dx+dy}$.

6. $\dfrac{2a^2-18a-72}{a^3+3a^2-9a-27}$. 7. $\dfrac{x^3+6x^2-x-30}{x^3+7x^2+2x-40}$.

8. $\dfrac{4x^4-2x^2-6}{4x^5+8x^4-2x^3-4x^2-6x-12}$.

9. $\dfrac{x^2+ax+bx+ab}{x^2+3ax+2a^2}$. 10. $\dfrac{a^3-a^2-7a+3}{a^4+2a^3+2a-1}$.

11. $\dfrac{20x^3+41x^2-10x-24}{24x^3+38x^2-26x-12}$.

12. $\dfrac{x^4+2x^3-8x^2-13x+6}{x^4-7x^2+1}$.

13. $\dfrac{x^2 - x - 6}{x^3 + 6x^2 + 12x + 8}$. 14. $\dfrac{x^6 + y^6}{x^4 - y^4}$.

15. $\dfrac{x^{12} + y^{12}}{x^8 - y^8}$. 16. $\dfrac{a^4 + a^2x^2 + x^4}{a^6 - x^6}$.

17. $\dfrac{3a^4 - 14a^3 - 9a + 2}{2a^4 - 9a^3 - 14a + 3}$.

18. $\dfrac{a^2 - (x - y)^2}{a^2 - ax + 2ay - xy + y^2}$.

19. $\dfrac{8a^3 + 36a^2x + 54ax^2 + 27x^3}{4a^2 + 8ax + 3x^2}$.

20. $\dfrac{6x^2 + 7x - 3}{18x^2 - 12x + 2}$. 21. $\dfrac{x^5 - y^5}{x^6 - y^6}$.

22. $\dfrac{x^3 - 5x^2 - 2x + 24}{x^3 - 4x^2 - 3x + 18}$. 23. $\dfrac{2x + bx - 3b - 6}{ax - 3a + bx - 3b}$.

24. $\dfrac{2ax + 3bx + 4a + 6b}{x^2 + x(b + 2) + 2b}$.

25. $\dfrac{x^2 - (a + b)^2}{a^2 - (x + b)^2}$. 26. $\dfrac{(x + y)^2 - (x - y)^2}{(x - y)^2 - (x + y)^2}$.

87. REDUCTION OF FRACTIONS TO INTEGRAL OR MIXED FORMS.

Reduce to Integral or Mixed Forms:

1. $\dfrac{a^4 - 3a^3 + 2a^2 + 1}{a + 1}$. 2. $\dfrac{x^4 - a^4}{x^3 - a^3}$.

3. $\dfrac{3x^4 - 23x^3 - 5x^2 + 80x + 50}{x^2 - 6x - 10}$.

4. $\dfrac{x^3 - x^2 - x - 1}{x - 1}$. 6. $\dfrac{3x^5 + x^4 + 2x + 1}{3x^2 + 1}$.

5. $\dfrac{x^3 + x^2 + x + 1}{x + 1}$. 7. $\dfrac{6x^2 - 19x + 15}{2x - 3}$.

8. $\frac{15x^2 - 11x + 6}{5x - 2}$.

9. $\frac{8x^2 - 6x - 3}{x - 1}$.

10. $\frac{56x^2 + 23x + 4}{8x + 1}$.

11. $\frac{x^6 - y^6 - z^6}{x^2 - y^2}$.

12. $\frac{2x^2 + 3x + 2}{2x + 1}$.

88. REDUCTION OF MIXED TO FRACTIONAL FORMS.

Reduce to Fractional Forms:

1. $\frac{a + b}{a} + 2$.

2. $a + b - \frac{a + b}{2}$.

3. $2a - b + \frac{a^2 - b^2}{2a + b}$.

4. $2x^2 - \frac{2x(1 + xy)}{x + y}$.

5. $\frac{(x + y)^2}{xy} - 4$.

6. $2a - 7 - \frac{2a^2 - a - 20}{a + 3}$.

7. $\frac{3x^2 - 6y}{x + 3} - 3x + 2y$.

8. $a^2 - 2ay + y^2 - \frac{x^2(a^2 + y^2) - a^2y^2}{x^2 - y^2}$.

9. $2x + y + 1 - \frac{2x^2 + 3xy + y^2 + x + y}{x + y}$.

10. $\frac{3x^2 - xy + 2y^2}{x - y} - 3x + 2y - 6$.

11. $\frac{6a^2 + 5a + 1}{2a + 1} - 3a - 1$.

12. $x + 3 + \frac{x^2 + 9}{x - 3}$.

13. $x + 2 - \frac{x^2 + x - 6}{x - 3}$.

14. $x + 2a - \frac{x^2 - x + 2a^2}{x + a}$.

89. ADDITION AND SUBTRACTION OF FRACTIONS.

Combine the following fractions:

1. $$\frac{1}{x^2-3x+2}+\frac{2}{x^2-4x+3}+\frac{4}{x^2-5x+6}.$$

2. $$\frac{x-3}{2x^2-5x+2}+\frac{x+2}{2x^2-7x+3}-\frac{2x+1}{x^2-5x+6}.$$

3. $$\frac{x}{3x^2-7x+2}-\frac{2}{3x^2-5x-2}+\frac{3}{9x^2-1}.$$

4. $$\frac{2a}{a^2-b^2}-\frac{a^2}{a^3+b^3}-\frac{1}{a-b}.$$

5. $$\frac{1}{6x+6}+\frac{1}{6-6x}-\frac{1}{3x^2-3}.$$

6. $$1-\frac{x-y}{x+y}+\frac{2y^2}{x^2-y^2}+\frac{2xy}{x^2+y^2}.$$

7. $$\frac{a-b}{2x-4-xy+2y}+\frac{x+2}{2a-ay+2b-by}.$$

8. $$\frac{1}{x-1}+\frac{x+1}{x^2-3x+2}+\frac{x-2}{2x^2-5x+2}.$$

9. $$\frac{2}{9x^3-x-2}+\frac{2x-3}{3x^3-10x^2-7x-4}+\frac{1}{x-4}.$$

10. $$\frac{x-3}{x^2-2x-3}+\frac{x-2}{x^2-3x+2}+\frac{1}{x^2-1}.$$

11. $$\frac{1}{x-2}+\frac{2}{x^2-4x+4}+\frac{x^2+4}{x^3-6x^2+12x-8}.$$

12. $$\frac{1}{x^2-(x+y)^2}+\frac{1}{x^2-(x-y)^2}+\frac{1}{4x^2-y^2}.$$

13. $$\frac{1}{3x^3-27x}+\frac{-1}{x^2+x-12}+\frac{-2}{(x+3)^2}.$$

14. $$\frac{x+3}{x^2-9}+\frac{x-3}{x^3-27}+\frac{2}{x-3}+\frac{5}{x^2+3x+9}.$$

15. $\dfrac{x-1}{x^2+3x+2}+\dfrac{x+2}{x^2-1}+\dfrac{x+3}{x^2-2x-3}$

16. $\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}+\dfrac{a^2+b^2}{a^2-b^2}.$

17. $\dfrac{2}{x^2+3x+2}+\dfrac{2}{x^2+5x+6}+\dfrac{2}{x^2+4x+3}.$

18. $\dfrac{3}{x+a}-\dfrac{1}{x+3a}+\dfrac{3}{a-x}+\dfrac{1}{x-3a}.$

19. $\dfrac{x+2}{x^2-3x+2}-\dfrac{x-1}{x^2-5x+6}-\dfrac{1}{x^2-4x+3}.$

20. $\dfrac{x+2}{(x-2)(x-3)}-\dfrac{x+2}{(3-x)(x-4)}+\dfrac{1}{(4-x)(2-x)}.$

21. $\dfrac{5a+2}{(a-1)(3-a)}-\dfrac{2a+3}{(2-a)(a-1)}+\dfrac{3a-2}{(a-3)(a-2)}.$

22. $\dfrac{2}{(x+3)(x-2)}+\dfrac{3}{(3-x)(x+2)}-\dfrac{1}{(2-x)(x-3)}.$

23. $\dfrac{x+5}{(x-1)(x+2)}+\dfrac{2(x+2)}{(x+5)(1-x)}-\dfrac{1-x}{(x+2)(x+5)}.$

24. $\dfrac{2}{x-3a}+\dfrac{2a}{(x-2a)^2}-\dfrac{x-a}{(3a-x)(2a-x)}.$

25. $\dfrac{a+5}{(2-a)(3-a)}-\dfrac{a+2}{(3-a)(a-5)}+\dfrac{a+3}{(a-2)(a-5)}.$

26. $\dfrac{x-6}{2(x^2-9x+18)}+\dfrac{x-5}{(x-5)(2-x)}+\dfrac{1}{2(x-1)}.$

27. $\dfrac{x(16-x)}{x^2-4}+\dfrac{2x+3}{2-x}-\dfrac{2-3x}{x+2}.$

28. $\dfrac{5}{3-6x}-\dfrac{5}{3+6x}-\dfrac{x}{2+8x^2}.$

29. $\dfrac{1}{4(1+x)}+\dfrac{1}{4(1-x)}+\dfrac{1}{2(1+x^2)}.$

30. $$\frac{2x}{x^2+x-2}-\frac{2x}{x^2+2x-3}+\frac{1}{2x^2-x-1}.$$

31. $$\frac{3-2b}{2-2x-a+ax}+\frac{2+a}{3-3x-b+bx}.$$

32. $$\frac{2x+y}{4x^2-12xy+9y^2}-\frac{4x^2-10xy+9y^2}{8x^3-36x^2y+54xy^2-27y^3}.$$

33. $$\frac{a^3-b^3}{a^3+2a^2b-ab^2-2b^3}+\frac{a^3+b^3}{a^3-2a^2b-ab^2+2b^3}.$$

34. $$\frac{x-2y}{3x^2-5xy+2y^2}-\frac{x+3y}{4x^2-9xy+5y^2}$$
$$+\frac{x+y}{12x^2-23xy+10y^2}.$$

35. $$\frac{2a-7b}{2a^2-5ab-7b^2}+\frac{4a-5b}{4a^2-ab-5b^2}$$
$$+\frac{2a-7b}{8a^2-38ab+35b^2}.$$

36. $$\frac{x+2}{x^3-10x^2+31x-30}-\frac{x-2}{x^3-9x^2+26x-24}.$$

37. $$\frac{1}{3xy^2-3x^3+x^2y-y^3}+\frac{1}{3x^3-3x^2y+xy^2-y^3}.$$

38. $$\frac{a-5}{a^3-9a^2+26a-24}-\frac{a+6}{a^3-6a^2+11a-6}.$$

39. $$\frac{a-5}{a^3-10a^2+31a-30}-\frac{a-3}{a^3-9a^2+26a-24}.$$

40. $$\frac{x-b}{x^2+(a+c)x+ac}-\frac{x-a}{x^2+(b+c)x+bc}.$$

90. MULTIPLICATION OF FRACTIONS.

Simplify the following:

1. $$\frac{a^4-x^4}{a^3-x^3}\times\frac{a^2-x^2}{a^3+x^3}\times\frac{a^6-x^6}{a^2+x^2}.$$

2. $\frac{x^2+4x-5}{x^2-2x-3}\times\frac{x^2-x-6}{x^2-4x-5}\times\frac{x^2+2x+1}{x^2+x-2}$.

3. $\left(x-\frac{y^2}{x}\right)\left(\frac{x}{y}+\frac{y}{x}\right)\times\frac{xy}{x^4-y^4}$.

4. $\frac{a^2-x^2}{a+y}\times\frac{a^2-y^2}{ax+x^2}\times\left(a+\frac{ax}{a-x}\right)$.

5. $\frac{x^2-(x-y)^2}{x^2-(x+y)^2}\times\frac{4x^2+4xy+y^2}{6x^2+xy-2y^2}\times\frac{3x^2-xy-2y^2}{2x+y}$.

6. $\frac{3a^2-a-2}{2a^2-a-1}\times\frac{9a^2+12a+4}{a^2+2a-3}\times\frac{2a^2+7a+3}{3a^2+8a+4}$.

7. $\frac{9x^2-6x}{4x^2-8x+3}\times\frac{2x^2+3x-9}{6x^2-7x+2}\times\frac{2x^2+13x-7}{2x^2+6x}$.

8. $\left(4x^2+14x+\frac{98x-27}{2x-7}\right)\left(\frac{1}{6}-\frac{3x+29}{12x^2+18x+27}\right)$.

9. $\frac{a^2-b^2}{a^3-b^3}\times\frac{(a-b)^2}{(a+b)^2}\times\frac{a^4-b^4}{a^3-b^3}\times\frac{a^2+ab+b^2}{(a^2-b^2)^2}$.

10. $\frac{x^2+5xy+6y^2}{x^2-y^2}\times\frac{ax+bx+ay+by}{ax+3by+bx+3ay}$
$\times\frac{2(x-y)}{5a(x+2y)}$.

11. $\frac{x^2-y^2}{x^2-3xy+2y^2}\times\frac{xy-2y^2}{x^2+xy}\times\frac{x^2-xy}{(x-y)^2}$.

12. $\frac{x^2-2ax+a^2}{x^2+4ax-5a^2}\times\frac{x^2-9a^2}{ax+2a^2}\times\frac{x^2+5ax}{x^2-4ax+3a^2}$.

13. $\left(b+\frac{ab}{b-a}\right)\left(b-\frac{ab}{a+b}\right)\left(\frac{b^2-a^2}{b^2+a^2}\right)$.

14. $\frac{a^5-x^5}{a-x}\times\frac{a+x}{(a-x)^2}\times\frac{a^2-x^2}{a^2+x^2}\times\frac{a^4-x^4}{(a+x)^3}$.

15. $\frac{a^4-x^4}{a^3-x^3}\times\frac{a^2-x^2}{a^3+x^3}\times\frac{a^4+a^2x^2+x^4}{a^2+x^2}$.

91. MULTIPLICATION AND DIVISION OF FRACTIONS.

Simplify the following:

1. $$\frac{x^2-x-6}{x^2+x-6}\div\frac{x^2+4x+4}{x^2+2x-3}.$$

2. $$\frac{4x^2-1}{6x^2-13x+6}\div\frac{2x^2+3x+1}{2x^2-x-3}.$$

3. $$\frac{x^2-3x+2}{x^2-7x+12}\div\frac{x^2+x-2}{x^2-x-6}.$$

4. $$\left(3x-5-\frac{2}{x}\right)\left(3x+5-\frac{2}{x}\right)\div\left(x-\frac{4}{x}\right).$$

5. $$\frac{xy-2y}{x^2-x}\times\frac{x^2+xy}{xy-y^2}\div\frac{xy+y^2}{x^2-xy}.$$

6. $$\frac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc}\div\frac{x^2-a^2}{x^2-b^2}.$$

7. $$\frac{x^2-5x+6}{x^2+x-2}\times\frac{x^2+3x-4}{ax+bx-3a-3b}$$
$$\div\frac{x^2+2x-8}{ax-a+bx-b}.$$

8. $$\left(\frac{x^4-a^4}{x^2-2ax+a^2}\div\frac{x^2+ax}{x-a}\right)\times\frac{x^5-a^2x^3}{x^3+a^3}$$
$$\div\left(\frac{x}{a}-\frac{a}{x}\right).$$

9. $$\left(\frac{x+y}{y}+\frac{y}{x+y}\right)\div\frac{x+y}{xy}.$$

10. $$\frac{a-1}{2}\times\frac{1+a+a^2}{a}\div\frac{a^3-1}{a^2}.$$

11. $$\frac{(a+b)^2-4ab}{(a-b)^2+4ab}\div\frac{a^2-2ab+b^2}{a^2-b^2}.$$

12. $$\left(2-\frac{a+b}{a-b}\right)\div\left(2+\frac{a+b}{a-b}\right).$$

13. $\dfrac{a^3-b^3}{a^2-b^2}\times\dfrac{a^4-b^4}{a-b}\div\dfrac{a^2+b^2}{a^3+b^3}$.

14. $\dfrac{2(a-x)}{a(a^2+2a)}\times\dfrac{a+b}{a-b}\div\dfrac{a^2+ab-ax-bx}{a^2+2a-ab-2b}$.

15. $\left(\dfrac{2}{x}-\dfrac{1}{a+x}+\dfrac{1}{a-x}\right)\div\left(\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right)$.

92. COMPLEX FRACTIONS.

Simplify the following:

1. $\dfrac{\frac{a}{2}-\frac{b}{3}}{\frac{a}{2}+\frac{b}{3}}$.

2. $\dfrac{1-\frac{y^2}{x^2}}{1+\frac{y^2}{x^2}}$.

3. $\dfrac{\frac{1}{x}-\frac{2}{x^2}-\frac{3}{x^3}}{\frac{9}{x}-x}$.

4. $\dfrac{\frac{a}{x^2}+\frac{x}{a^2}}{\frac{1}{a^2}-\frac{1}{ax}+\frac{1}{x^2}}$.

5. $\dfrac{2x+\frac{x}{x-2}}{2x-\frac{x}{x-2}}$.

6. $\dfrac{4x^2-16x+15}{\frac{(2x-3)^2}{4}-1}$.

7. $\dfrac{\frac{1}{2x-1}-\frac{1}{2x+1}}{4-\frac{1}{x^2}}$.

8. $\dfrac{\frac{x^2+10x+21}{x^2-2x-15}}{\frac{2x^2+x-15}{2x^2-19x+35}}$.

9. $\dfrac{\frac{a^3+b^3}{a^2-b^2}}{\frac{a^2+ab}{a^2-ab}}$.

10. $\dfrac{2+\left(\frac{a-x}{a+x}\right)^2}{2-\left(\frac{a-x}{a+x}\right)^2}$.

11. $\dfrac{\frac{12x^2+x-1}{15x^2+8x+1}}{\frac{4x^2-5x+1}{12x^3-12x^2+4x-4}}$.

12. $\dfrac{\frac{x+y}{x-y}-\frac{x-y}{x+y}}{1-\frac{x^2+y^2}{(x+y)^2}}$.

13. $$\frac{\dfrac{1}{x^2-9x+20}+\dfrac{1}{x^2-11x+30}}{\dfrac{2}{x^2-10x+24}}.$$

14. $$\frac{\dfrac{1}{x^2-7x+12}-\dfrac{1}{x^2-5x+6}}{\dfrac{x-1}{x^2-7x+12}}.$$

15. $$\frac{\dfrac{x^2+ax+bx+ab}{x^2+ax+cx+ac}}{\dfrac{x^2-ax-bx+ab}{x^2-ax+cx-ac}}.$$

16. $$\frac{\dfrac{\dfrac{2x}{3}-y}{2}+\dfrac{\dfrac{3x}{4}-\dfrac{6y}{5}}{3}+\dfrac{y-8x}{9}}{\dfrac{x+y}{4}-\dfrac{x-y}{9}+\dfrac{1}{5}}.$$

17. $$\frac{\dfrac{5x^2}{6}-\dfrac{3x}{2}-\dfrac{\dfrac{x^2}{4}-x}{3}-\dfrac{x^2+\dfrac{3x}{2}-\dfrac{5}{3}}{6}}{\dfrac{x^2-5x+6}{12}-\dfrac{x^2+5x+6}{9}+\dfrac{x^2+18x+3}{18}}.$$

18. $$\frac{\dfrac{(xy-3y^2)^2}{x^3y^2-27y^5}}{\dfrac{x^2+xy-2y^2}{x^3-y^3}}\times\frac{x+2y}{(x-3y)(x^3-y^3)}.$$

19. $$\frac{\dfrac{x^2-y^2}{2}-\left(\dfrac{1}{2x+1}+\dfrac{1}{2x-1}+\dfrac{4x}{1-4x^2}\right)}{\dfrac{x+y}{x-y}}.$$

20. $$\frac{2+\dfrac{x^2-5x+6}{x^2-7x+12}}{3-\dfrac{x^2-9x+20}{x^2+6x-55}}.$$

21. $$\frac{\dfrac{(a^3-b^3)(a+b)}{(a^3+b^3)(a-b)}}{\dfrac{a^4-b^4}{(a^3-b^3)(a+b)}}\times\frac{a^2-ab+b^2}{a^2+ab+b^2}.$$

22. $$\frac{\dfrac{xy-y^2}{x^2-xy}+\dfrac{x^2-y^2}{xy}}{\dfrac{x(x+y)}{y(x-y)}}.$$

23. $$\frac{\dfrac{1}{2x^2-x-1}-\dfrac{1}{2x^2+x-3}}{\dfrac{2}{4x^2+8x+3}}.$$

24. $$\frac{\dfrac{x^6-1}{x^2+1}\div(x^2-x+1)}{\dfrac{x^3-x^2-7x+3}{x^4+2x^3+2x-1}}.$$

25. $$\frac{\dfrac{2ax}{x^2-a^2}+\dfrac{3a}{a+x}+\dfrac{a}{a-x}}{\dfrac{a-x}{2a^2+3ax+x^2}}\times\frac{a^2-x^2}{4a}.$$

26. $$\frac{\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4y^2}{x^2-y^2}\right)\left(\dfrac{x+y}{2y}\right)}{\dfrac{x^2-y^2}{x^2+xy-2y^2}\div\dfrac{x+y}{x-y}}$$

27. $$\frac{x^5-\dfrac{1}{x^5}}{x-\dfrac{1}{x}}.$$

28. $$\frac{1+\dfrac{1}{x^2}+\dfrac{1}{x^4}}{1+\dfrac{1}{x}+\dfrac{1}{x^2}}.$$

29. $$\frac{\dfrac{mx+ma-nx-na}{mx-nx-bm+bn}}{\dfrac{x^2-a^2}{x^2-b^2}\times\dfrac{x^2+x(a+b)+ab}{x^2-x(a+b)+ab}}.$$

30. $$\frac{1 - x + x^2 - x^3 + \dfrac{x^4}{1 + x}}{\dfrac{6 + 5x + x^2}{2 + 3x + x^2}}.$$

31. $$\frac{\dfrac{y - [y - (x + y) - \{y - (y - \overline{x - y})\}]}{x(a + b) + ay + by}}{\dfrac{(a + b)(x - y)}{x^2 - y^2}}.$$

32. $$\frac{\left(\dfrac{x}{x - y} - \dfrac{y}{x + y}\right) \div \left(\dfrac{x^2}{x^2 + y^2} + \dfrac{y^2}{x^2 - y^2}\right)}{\dfrac{(x^2 + y^2)^2}{(x^3 - y^3)(x^2 - y^2)}}.$$

33. $$\frac{\dfrac{x^2 + 4x - 5}{x^2 - 2x - 3} \times \dfrac{x^2 - x - 6}{x^2 - 4x - 5} \times \dfrac{x^2 + 2x + 1}{x^2 + x - 2}}{\dfrac{x^2 + 10x + 25}{x^2 - 25} \times \dfrac{x^2 + 2x - 8}{x^2 - 5x + 6} \div \dfrac{x^2 + 5x + 4}{x^2 - 5x + 6}}.$$

34. $$\frac{\left(x + 2 + \dfrac{4}{x - 3}\right)\left(x - 2 - \dfrac{4}{x + 1}\right)}{\dfrac{x^2 - x - 2}{x^2 - 5x - 6}}.$$

35. $$\frac{\left(1 + \dfrac{a}{b}\right)\left(1 - \dfrac{a}{b}\right) \div \dfrac{a - b}{a(a + b)}}{\dfrac{a^2 - b^2}{a - b} \times \dfrac{a + b}{b - a}}.$$

36. $$\frac{\dfrac{x^4 - 2x^2y^2 + y^4}{x^3y + xy^3} \div \dfrac{x^2 - y^2}{x^2 + y^2}}{\dfrac{x^2 - y^2}{4x^2y^2}}.$$

93. GENERAL EXERCISES IN FRACTIONS.

Simplify the following:

1. $$\frac{x^2-2y^2}{x^3+y^3}+\frac{y}{x^2-xy+y^2}+\frac{1}{x+y}.$$

2. $$\frac{2}{x-y}+\frac{x-y}{x^2+xy+y^2}-\frac{x^2+y^2}{x^3-y^3}.$$

3. $$\frac{\dfrac{x-2}{x^2-x-12}-\dfrac{x}{x^2+6x+9}}{\dfrac{6x^2+7x+2}{x^2-x-12}}.$$

$x = -2$

$x = -1$

4. $$\frac{x^3+8x^2+17x+10}{x^3+11x^2+35x+25}.$$

5. $$\frac{\dfrac{1}{x^2+3xy+2y^2}}{\dfrac{x}{x+y}-\dfrac{y}{x-2y}+\dfrac{2y}{x+2y}}.$$

6. $$\frac{x}{y}-\frac{(x^2-y^2)a}{y^2}+\frac{a^2x(x^2-y^2)}{y^2(y+ax)}.$$

7. $$\left(y-\frac{a^2-xy}{y-x}\right)\left(x+\frac{a^2-xy}{y-x}\right)+\left(\frac{a^2-xy}{y-x}\right)^2.$$

8. $$\frac{\dfrac{(1-2x)^2+(1+2x)^2}{(1-4x^2)-(1-2x)^2}}{\dfrac{(1+2x)^2-(1-4x^2)}{(1-2x)^2-(1+2x)^2}}.$$

9. $$\frac{x-2y-\dfrac{3x^2-4xy-x}{3x-1}}{\dfrac{1-x^2}{9x^2-6x+1}}\times\frac{1+x}{3x-1}.$$

10. $$\frac{\dfrac{x^2+y^2+2xy-z^2}{4x^2y^2-(x^2+y^2-z^2)^2}}{\dfrac{1}{x^2+y^2+z^2+2xz-2xy-2yz}}.$$

11. $\dfrac{a^3 - 3a^2 + 3a - 2}{a^3 - 4a^2 + 6a - 4}$.

12. $\dfrac{x-3}{x - 3 - \cfrac{\frac{x}{x-2}}{\frac{x-1}{x+3}}}$.

13. $2 + \cfrac{1}{3 - \cfrac{1}{1 - \cfrac{x}{y}}}$.

14. $\left(1 - \dfrac{2xy}{x^2 + y^2}\right)\left(\dfrac{(x+y)^2 + (x-y)^2}{(2x-y)^2 - y^2}\right)$.

15. $\left(12a^3 - 5a^4 + \dfrac{1}{6} - \dfrac{a}{3} - \dfrac{31a^2}{6}\right) \div \left(\dfrac{1}{6} + \dfrac{a^2}{2} - a\right)$.

16. $\dfrac{\dfrac{1}{x^2 - 5x + 6} - \dfrac{2}{x^2 - 4x + 3} + \dfrac{2}{x^2 - 3x + 2}}{\dfrac{1}{x(x-1)} - \dfrac{2}{1 - x^2} + \dfrac{1}{x(x+1)}}$.

17. $\dfrac{\dfrac{x^{12} - y^{12}}{x^3 - y^3} + (x^6 + y^6)(x^3 + y^3)}{2(x^2 + y^2)(x + y)}$.

18. $\dfrac{\dfrac{x^3 - y^3}{x + y} \times \dfrac{x^4 + x^2y^2 + y^4}{x^3 + y^3} \times \dfrac{x^2 - y^2}{x^2 + xy + y^2}}{x^6 - y^6}$.

19. $\dfrac{5a}{2} - \dfrac{21b}{2} - 3\left[\dfrac{b}{6} - 2\left\{\dfrac{a}{3} - b - 3\left(\dfrac{2a}{9} - \dfrac{5b}{6}\right)\right\}\right]$.

20. $\dfrac{\dfrac{x}{y} + \dfrac{y}{x}}{\dfrac{x}{y} - \dfrac{y}{x}} \div \dfrac{\dfrac{x^2}{y^2} - \dfrac{y^2}{x^2}}{\left(\dfrac{1}{y} + \dfrac{1}{x}\right)^2}$.

21. $\dfrac{x - (y - z) - [x - y - z - 2(y + z)]}{2(x^2 + 5xz + 6z^2)}$
$\times \dfrac{x^2 + xz - 6z^2}{x - 2z}$.

94. REDUCTION OF FRACTIONS HAVING NEGATIVE EXPONENTS.

1. Prove that $a^{-2} = \frac{1}{a^2}$.

2. What is the statement of the principle involved in Exercise 1?

3. What operation in the reduction of fractions having factors with negative exponents is established by this principle ?

MENTAL EXERCISES.

4. Clear of negative exponents

$$\frac{a^2b}{c^{-1}},\ \frac{a^{-1}b^{-1}}{2},\ \frac{abc^{-2}}{c^{-1}},\ \frac{a^{-1}b^{-1}c^2}{a^2b^2c^{-2}}.$$

5. Change to integral forms

$$\frac{2\,ab}{c},\ \frac{4\,a^2b}{2\,ab^{-1}},\ \frac{a^{-2}b^{-2}}{a^2b^{-1}c},\ \frac{abc}{(abc)^{-1}},\ \frac{a^2-b^2-c^2}{3^{-2}\,ab},$$

$$\frac{2\,a+b}{(2\,a-b)^{-1}},\ \frac{2\,ab-c}{a^{-2}}.$$

6. Express with positive exponents

$$2\,xy^{-1}+3\,x^{-2}y^{-2}+y^{-1},\ 5\,x^2y^{-2}+2^{-1}\,xy,$$

$$5\,xyz^{-1}+4\,x^{\frac{1}{3}}+y^{-\frac{1}{2}},\ \frac{2\,x^{-1}y^{-1}}{x^2y^2}.$$

GENERAL EXERCISES.

Clear of negative exponents and simplify the results in the following:

7. $\dfrac{a^{-1}+b^{-1}}{a}$.

8. $\dfrac{2\,a^{-2}+b}{a^{-2}\,b^2}$.

9. $\dfrac{x^3-\dfrac{1}{a^{-3}}}{\dfrac{1}{x^{-2}}-\dfrac{4}{a^{-2}}}\times\dfrac{x+\dfrac{2}{a^{-1}}}{\dfrac{1}{x^{-1}}-\dfrac{1}{a^{-1}}}$.

10. $\dfrac{\dfrac{1}{x^{-2}}-\dfrac{1}{x^{-1}}+\dfrac{x-1}{x+1}}{\dfrac{1}{x^{-1}}+(x+1)^{-1}}$.

11. $\dfrac{xy^{-1}+yx^{-1}+2}{x+y}+\dfrac{xy^{-1}+yx^{-1}-2}{x-y}.$

12. $\dfrac{x^2y^{-3}-x^{-1}}{xy^{-2}+y^{-1}+x^{-1}}.$

13. $\dfrac{1+\left(\dfrac{a+x}{a-x}\right)^{-2}}{1-\left(\dfrac{a+x}{a-x}\right)^{-2}}.$

14. $\dfrac{\left(\dfrac{1}{x^{-3}}+\dfrac{1}{y^{-3}}\right)^{-1}(x^2-xy+y^2)}{\left(\dfrac{1}{x^{-3}}-\dfrac{1}{y^{-3}}\right)^{-1}(x^2+xy+y^2)}.$

15. $\dfrac{1+\left(\dfrac{a+x}{a-x}\right)^{-1}}{1-\left(\dfrac{a+x}{a-x}\right)^{-1}}\div\dfrac{1+\left(\dfrac{a^2+x^2}{a^2-x^2}\right)^{-1}}{1-\dfrac{a^2-x^2}{a^2+x^2}}.$

16. $\dfrac{a^{-2}-b^{-2}}{a^{-3}-b^{-3}}.$

17. $\dfrac{a^{-2}\,b^{-2}}{a^{-2}+b^{-2}}\times\dfrac{a^2+b^2}{a^{-1}-b^{-1}}.$

18. $\dfrac{3\,a^{n+1}}{a^{n-1}\,x}\times\dfrac{x^{-5}yz^m}{a^2y^{-1}z^{m-1}}\times\dfrac{9\,z^2}{y^2z^3}.$

19. $\dfrac{x^{-1}y^{-1}z^{-n}}{ax^{-2}y^{-3}}\times\dfrac{a^nb^{n-1}c^{-n}}{x^{-3}yz}\times\dfrac{a^{-n}bc^{n-1}}{x^2yz^{1-n}}.$

20. $\dfrac{x^3+x^{-3}-5\,(x+x^{-1})}{x+x^{-1}}.$

21. $\frac{1}{4}\,(xa^{-1}-ax^{-1})\left(\dfrac{a^{-1}-x^{-1}}{a^{-1}+x^{-1}}-\dfrac{a^{-1}+x^{-1}}{a^{-1}-x^{-1}}\right).$

22. $\dfrac{(a^2b^{-2}+b^2a^{-2}+1)\,(a^2b^{-2}+b^2a^{-2}-1)}{a^4b^{-4}+b^4a^{-4}+1}.$

CYCLIC ORDER.

95. If the letters a, b, c are placed upon the circumference of a circle in the given order and the direction indicated by that order is constantly followed, they will be found in the groups abc, bca, cab, in which b follows a, c follows b, and a follows c.

The letters of these groups are said to be written in **Cyclic Order**.

96. It will be seen that if these groups are arranged cyclically in the given order, each can be derived from the one next preceding it by changing a into b, b into c, and c into a. Thus, any one of the groups being given, the others can be found by what is called a *cyclic change* of letters.

97. The observance of cyclic order is often important in algebraic work, especially in exercises of the general character of those in § 99, where it will be found that after the work has been reduced to proper cyclic order and one new numerator found, the remaining new numerators can be written by inspection, by the principle of § 96.

The foregoing principles can be extended to more than three letters, and to expressions which contain letters not belonging to the circular set.

98. MENTAL EXERCISES.

Change the following to the cyclic order:

1. $b - a$, $c - b$, $a - c$. 2. a^2cb, cb^2a, bac^2.

3. $a^2(c - b)$, $c^2(b - a)$, $b^2(a - c)$.

4. $(b - a)(a - c)$, $(c - b)(b - a)$, $(a - c)(c - b)$.

Complete the following series:

5. $a^2(b - c)$, . . . 7. $(b + c)(b + c - 2a)$, . . .

6. $b^2c^2(a - b)$, . . . 8. $a^4(b^2 - c^2 + a^2)$, . . .

99. APPLICATION TO FRACTIONS.

Combine the following fractions, observing the cyclic order in all operations:

1. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}.$

2. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}.$

3. $\dfrac{a}{(a-b)(a-c)}-\dfrac{b}{(a-b)(b-c)}+\dfrac{c}{(c-a)(c-b)}.$

4. $\dfrac{bc(x+a)}{(a-b)(a-c)}+\dfrac{ca(x+b)}{(b-c)(b-a)}+\dfrac{ab(x+c)}{(c-a)(c-b)}.$

5. $\dfrac{a^2-bc}{(c-a)(a-b)}+\dfrac{b^2-ca}{(a-b)(b-c)}+\dfrac{c^2-ab}{(b-c)(c-a)}.$

6. $\dfrac{x^2yz}{(x-y)(x-z)}+\dfrac{y^2xz}{(y-z)(y-x)}+\dfrac{z^2xy}{(z-x)(z-y)}.$

7. $\dfrac{1+a}{(a-b)(a-c)}+\dfrac{1+b}{(b-c)(b-a)}+\dfrac{1+c}{(c-a)(c-b)}.$

8. $\dfrac{b-c}{a^2-(b-c)^2}+\dfrac{c-a}{b^2-(c-a)^2}+\dfrac{a-b}{c^2-(a-b)^2}.$

9. $\dfrac{a^3}{(a-b)(a-c)}-\dfrac{b^3}{(c-b)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}.$

10. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}$

$+\dfrac{1}{c(c-a)(c-b)}.$

100. SUBSTITUTION.

1. If $x=\dfrac{a+b}{2}$, find the value of $\left(\dfrac{x-a}{x-b}\right)^3$

$-\dfrac{x-2a+b}{x+a-2b}.$

2. If $x = \dfrac{b^2 + c^2 - a^2}{2\,bc}$ and $y = \dfrac{a^2 - (b - c)^2}{(b + c)^2 - a^2}$,

find the value of $\dfrac{x + y}{1 - xy}$.

3. If $x = \dfrac{b - c}{a}$, $y = \dfrac{c - a}{b}$, and $z = \dfrac{a - b}{c}$,

find the value of $xyz + x + y + z$.

4. If $s = \frac{1}{2}(a + b + c)$, show that

$(s - a)(s - b)(s - c) = s^3 - \dfrac{s}{2}(a^2 + b^2 + c^2) -$ abc; also that $\dfrac{1}{s - a} + \dfrac{1}{s - b} + \dfrac{1}{s - c} - \dfrac{1}{s}$ $= \dfrac{abc}{s(s - a)(s - b)(s - c)}$.

101. TEST EXAMINATIONS IN FRACTIONS.

A.

Simplify the following:

1. $\dfrac{x - 2}{x^2 + 5x + 6} + \dfrac{x + 1}{x^2 - 2x - 8} - \dfrac{x + 2}{x^2 - x - 12}$.

2. $\left[\dfrac{3}{1 + x} - \dfrac{2}{1 - x} - \dfrac{5x}{x^2 - 1}\right] \div \dfrac{1 + x^4}{1 - x^4}$.

3. $\dfrac{2\,x^3 + 21\,x^2 + 55\,x + 42}{3\,x^3 + 32\,x^2 + 87\,x + 70}$.

4. $2\,x - 3\,y + 2 + \dfrac{2\,x - 6\,y + 8}{x - 2}$.

5. $\dfrac{y}{2(x - y)} - \dfrac{y}{2(x + y)} + \dfrac{y^4}{x^2(y^2 - x^2)}$.

6. $\dfrac{x^2 - 6\,xy + 9\,y^2}{x^2 - 4\,xy + 4\,y^2} \div \left[\dfrac{x^2 - 9\,y^2}{x^2 - 4\,y^2} \div \dfrac{x^2 + xy - 6\,y^2}{x^2 - xy - 6\,y^2}\right]$.

7. $$\frac{bc}{(c-a)(a-b)}+\frac{ca}{(a-b)(b-c)}+\frac{ab}{(b-c)(c-a)}.$$

8. $$\left\{x^2-\frac{(x^2+y^2-z^2)^2}{4y^2}\right\}\div\left\{\left(\frac{x+z}{y}\right)^2-1\right\}.$$

B.

Simplify the following:

1. $$\frac{1}{x^2-y^2}+\frac{1}{(x+y)^2}-\frac{1}{(x-y)^2}-\frac{x+y}{x^3+x^2y-xy^2-y^3}.$$

2. $$\left(2-\frac{a+b}{a-b}\right)\left(2-\frac{a-b}{a+b}\right).$$

3. $$\frac{x^{12}-y^{12}}{x^4-y^4}\div(x^4+x^2y^2+y^4).$$

4. $$\frac{x^3-6x^2+5x+12}{x^3-7x^2+7x+15}.$$

5. $$\frac{\dfrac{2y-x}{x-y}+\dfrac{2x+y}{x+y}-\dfrac{x^2}{x^2-y^2}}{\dfrac{x^2+y^2}{x^2-y^2}}\times\frac{x^4-y^4}{x^2}.$$

6. $$\frac{b-c}{x^2-(b+c)x+bc}+\frac{c-a}{x^2-(c+a)x+ca}+\frac{a-b}{x^2-(a+b)x+ab}.$$

7. $$3-\cfrac{1}{2-\cfrac{1}{1-\cfrac{2x-\frac{1}{3}}{5x}}}.$$

8. $$\left\{1-\frac{(a^2+b^2-c^2-d^2)^2}{4(ab+cd)^2}\right\}\div\left\{\frac{(a+b)^2-(c-d)^2}{4(ab+cd)}\right\}.$$

XI. SIMPLE EQUATIONS.

DEFINITIONS.

102. An **Equation** is a statement of the equality of two algebraic expressions.

103. The **Members** of an equation are the expressions connected by the equality sign. The one on the left of this sign is called the *first* member; and the one on the right, the *second* member. The members are sometimes called the *sides* of the equation.

104. An **Identical Equation**, or an **Identity**, is one whose members are alike in form, or may be made so by proper reduction.

It will, therefore, be true for *all* values of its letters.

105. A **Conditional Equation** is one which is only true for *particular* values of its letters. This is the ordinary equation of Algebra.

106. A **Numerical Equation** is one which has its known quantities represented by figures.

107. A **Literal Equation** is one which has its known quantities represented by letters, or by both figures and letters.

108. The **Degree** of an equation of one unknown quantity is indicated by the highest exponent of that quantity in the equation. If the equation contains more than one unknown quantity, its degree is indicated by the *sum* of the exponents of the unknown quantities in the term where this sum is the greatest.

The degree of an equation cannot, in general, be determined until it has been cleared of fractions, radical signs, fractional and negative exponents.

109. A **Simple Equation** is an equation of the *first* degree; as,

$$2x + 3 = 15, \text{ or } 3x + 2y = 17.$$

110. To **solve** an equation is to find the values of its unknown quantities (one or more). These values are called the *roots* of the equation. Roots are *verified* when they reduce the equation to an identity upon substituting them for the letters they represent. The equation is then said to be *satisfied*.

SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

111. NUMERICAL EQUATIONS.

Solve the following equations:

1. $\frac{6x+7}{5} - \frac{2x-1}{10} = 4\frac{1}{2}$.

2. $\frac{8x-9}{14} - \frac{3x-5}{21} + \frac{2x-7}{3} = 0$.

3. $\frac{2x-5}{9} - \frac{x+3}{5} - \frac{x}{3} = 2x - 15\frac{1}{3}$.

4. $\frac{8x+37}{18} - \frac{7x-29}{5x-12} = \frac{4x+12}{9}$.

5. $\frac{7}{2x+6} - \frac{3\frac{1}{2}}{2x+2} = \frac{4}{x+3} - \frac{2}{x+1}$.

6. $\frac{8x+5}{14} - \frac{3-7x}{6x+2} = \frac{16x+15}{28} + \frac{2\frac{1}{4}}{7}$.

7. $\frac{6x+7}{15} - \frac{2x-2}{7x-6} = \frac{2x+1}{5}$.

8. $(3x-1)^2 + (4x-2)^2 = (5x-3)^2$.

9. $\frac{5x^2+x-3}{5x-4} = \frac{7x^2-3x-9}{7x-10}$.

10. $\frac{2}{2x+1} - \frac{3}{3x-1} = \frac{3}{3x-2} - \frac{2}{2x-3}$.

11. $\frac{x}{x-2} - \frac{x+1}{x-1} = \frac{x-8}{x-6} - \frac{x-9}{x-7}$.

12. $\frac{5}{x-2} - \frac{10}{2x+4} = \frac{2}{x-4} - \frac{2}{x+6}$.

13. $\frac{x+5}{x+4}+\frac{x-15}{x-16}=\frac{x-4}{x-5}+\frac{x-6}{x-7}.$

14. $\frac{x(x+3)}{(x+1)(x+2)}+\frac{4}{3x(x+2)}=1.$

15. $\frac{4x+15}{6}+\frac{\frac{2x}{5}-\frac{1}{2}}{6}-\frac{x+\frac{1}{4}}{\frac{1}{5}}=5\frac{13}{30}.$

16. $\frac{3(x+2)}{\frac{4}{3}}+\frac{\frac{1}{5}(6x-2)}{\frac{1}{7}}=\frac{\frac{2x}{3}+1}{\frac{1}{6}}+9.$

17. $\frac{3\frac{1}{4}-4x}{1\frac{1}{3}}-\frac{49}{54}(3\frac{1}{2}-5x)=\frac{7}{16}+\frac{55}{108}(3x-2).$

18. $\frac{2}{1-2x}-\frac{2}{2x-7}=1-\frac{4x^2-1}{4x^2-16x+7}.$

19. $\frac{x}{2}-\frac{\frac{1}{3}(2x-3)-\frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)}=\frac{3}{2}\left(\frac{x^2+2}{3x-1}\right).$

20. $(x-1)^3+(x-2)^3+(x-3)^3=$
$3(x-1)(x-2)(x-3).$

21. $\frac{x+2}{x}+\frac{x-7}{x-5}-\frac{x+3}{x+1}=\frac{x-6}{x-4}.$

22. $\frac{\frac{1}{8}(x-2)}{\frac{2}{3}}+\frac{\frac{3}{4}(6x-7)}{5}=\frac{\frac{2x}{3}-1}{\frac{1}{6}}-1\frac{1}{4}.$

23. $\frac{4}{x+2}+\frac{7}{x+3}=\frac{37}{x^2+5x+6}.$

24. $\frac{1}{x-2}-\frac{1}{x-4}=\frac{1}{x-6}-\frac{1}{x-8}.$

25. $4x-\{3x-(2x-\overline{6x+1})\}=4x-(5x-2)$
$-23.$

26. $3[4x-5-2(3x-4)+5\{2x-3-(2x$
$-7[x-5])\}]=39-x.$

27. $7x^2-[2x-\{2x^2-(3x-4)\}+3x]=9x^2+35$
$-[5x-(3x+7)].$

112. LITERAL EQUATIONS.

Solve the following equations:

1. $\frac{x-a}{x+a}=\frac{3b+x}{2b+x}.$

2. $\frac{x-a}{a-b}-\frac{x+a}{a+b}=\frac{2ax}{a^2-b^2}.$

3. $\frac{1}{a}-\frac{1}{x}=\frac{1}{x}-\frac{1}{b}.$

4. $\frac{x}{x-a}-\frac{x+a-b}{x-b}=\frac{a(a-b)}{(x-c)(x-d)}.$

5. $a^2(x-a)+b^2(x-b)=abx.$

6. $\frac{x-1}{x+1}=\frac{1-a}{1+a}.$

7. $(x+2a)(x-a)^2=(x+2b)(x-b)^2.$

8. $(x-a)(x-b)=(x-a-b)^2.$

9. $\frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{x-c}.$

10. $\frac{a}{x+a}+\frac{b}{x+b}=\frac{a+b}{x+c}.$

11. $\frac{x^2-a^2}{bx}-\frac{a-x}{b}=\frac{2x}{b}+\frac{a}{x}.$

12. $x(x-a)+x(x-b)=2(x-a)(x-b).$

13. $(x-a)^2-(x-b)^2=(a-b)^2.$

14. $\frac{x}{a}+\frac{x}{b-a}=\frac{a}{b+a}.$

15. $\frac{a+b}{x}+\frac{a^2-b^2}{a^2+ab}=\frac{a}{x}-\frac{b}{a}.$

16. $(a-b)(x-c)-(b-c)(x-a)-(c-a)(x-b)$
$=0.$

17. $\frac{a^2c}{b}+x^2=(a+x)(b+x)-a(b+c).$

18. $(x-a)^3(x+a-2b)=(x-b)^3(x-2a+b).$

19. $\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}-\dfrac{x^2}{a^2-x^2}=1.$

20. $\dfrac{x^2-(a+b)x+2ab}{x-a}=\dfrac{x^2-2bx+2b^2}{x-b}+\dfrac{b(a-b)}{x-c}.$

21. $\dfrac{x+7a}{x+6a}+\dfrac{x-a}{x-3a}+\dfrac{a-x}{x+2a}=\dfrac{x+7a}{x+a}.$

22. $3cx+\dfrac{bx}{a}=\dfrac{3abc}{a+b}+\dfrac{a^2b^2}{(a+b)^3}+\dfrac{(2a+b)b^2x}{a(a+b)^2}.$

23. $\dfrac{x^3+2a^3}{x+a}-\dfrac{x^3-2a^3}{x-a}=\dfrac{a^3}{x^2-a^2}-2ax.$

24. $\dfrac{(m-n)(x+a)}{x+b}+\dfrac{(a-b)(x+m)}{x+n}=a-b+m-n.$

25. $(x-a)^3+(x-b)^3+(x-c)^3=$
$3(x-a)(x-b)(x-c).$

26. $\dfrac{1}{x-a+b}+\dfrac{1}{x+a-b}=\dfrac{2}{x-a-b}.$

27. $(b-c)(x-a)^3+(c-a)(x-b)^3$
$+(a-b)(x-c)^3=0.$

113. PROBLEMS PRODUCING SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

1. The sum of two numbers is 52, and the greater exceeds the less by 4. What are the numbers?

2. The difference of two numbers is 5, and the difference of their squares is 55. What are the numbers?

3. The difference of two numbers is 36, and the greater is ten times the less. What are the numbers?

4. The sum of two numbers is 90, and $\frac{1}{5}$ of the greater is equal to $\frac{1}{4}$ of the less. What are the numbers?

5. Find a number which is as much above 50 as $\frac{1}{6}$ of the same number is below 20.

6. Ten times the difference between the fifth and sixth parts of a certain number is 40 less than the number. What is the number?

7. Divide 100 into two such parts that four times the greater part shall exceed five times the less by 22.

8. Five dollars, in dimes and quarters, were distributed among 26 boys. If each boy received one coin, how many received dimes?

9. Ten dollars were distributed among 5 men, 3 women, and 14 children, so that a man received as much as 2 women, and a woman as much as 2 children. How much did each child receive?

10. Into a cask $\frac{2}{3}$ full 10 gallons were poured when it was found to be $\frac{5}{6}$ full. How many gallons did the cask hold?

11. Each of two casks contains the same number of gallons. After drawing 10 gallons from one and pouring them into the other, the first contains $\frac{5}{6}$ as many as the second. How many gallons in each at first?

12. A man gave a certain number of boys a half-dime each, and had 75 cents remaining; but if he had given them a dime each as long as his money lasted, 5 would have received nothing. Find the number of boys and the amount of money.

13. Divide $600 among 5 persons so that the first and second together shall receive $150; the first and third, $250; the first and fourth, $200; and the fifth $100 less than the third. How much will each receive?

14. Into what two sums can $1000 be divided so that the income of one at 6 per cent shall be equal to the income of the other at 4 per cent?

15. There are three consecutive numbers. If $\frac{1}{2}$ of the first is added to $\frac{2}{3}$ of the second and from the sum $\frac{1}{5}$ of the third is subtracted, the remainder will be $12\frac{5}{6}$. What are the numbers?

16. A boy buys a certain number of apples at the rate of 3 for a cent, and $\frac{2}{3}$ as many more at the rate of 2 for a cent. If he sells the whole number at the rate of 4 for 5 cents and gains 68 cents, how many did he buy?

17. In a certain mixture there is twice as much wine as water; and 5 gallons more than $\frac{1}{2}$ of the whole is wine, and 5 gallons less than $\frac{1}{2}$ of the whole is water. How many gallons in the mixture?

18. A company of 500 soldiers has provisions for 60 days. After 12 days it is joined by another detachment and the provisions last only 40 days longer. How many men in the new detachment?

19. A boy is $\frac{1}{6}$ as old as his father and 5 years older than his sister. If the sum of the ages of all three is 51, how old is the father?

20. A boy having a certain number of apples gives to one companion one apple more than half of what he has, to another half an apple more than half of what remain, and to another half an apple more than half of what still remain, and has one left. How many had he at first?

21. The numerator of a certain fraction is 4 less than the denominator, but if 1 is added to each the fraction will be equal to $\frac{9}{10}$. What is the fraction?

22. Two-thirds of a man's property is invested at 5 per cent, $\frac{1}{6}$ at 4 per cent, and the remainder at 6 per cent. If the total income is $450, what is the value of the property?

23. The distance from A to B is 100 miles. A train leaving A at a certain rate meets with an accident 20 miles from B, reducing its speed one-half, and causing it to reach B an hour late. What was its regular rate per hour?

24. Two trains start from the same station going in the same direction, one at the rate of 20 miles per hour, and the other 40 minutes later at the rate of 30 miles per hour. If the faster train be delayed 30 minutes by an accident, at what distance from the starting-point will it overtake the the other?

25. A and B trade with equal capital. In the first year A doubled his capital and had $500 over, but in the same time B lost $500 less than $\frac{1}{2}$ of his capital. If A's gain was five times B's loss, what was the capital of each?

26. A certain number consists of two digits of which the one in units' place is four times the one in tens' place. If the order of the digits is inverted and 2 is added to the resulting number, the new number will be three times the original number. Find the original number.

27. A man distributing a number of pennies among some boys finds that he lacks 2 pennies when he attempts to give them 12 apiece, and has three remaining when he gives them 11 apiece. Find the number of boys and of pennies.

28. A number with three digits is composed of consecutive odd numbers. If the number is divided by the sum of its digits, the quotient will be 23 and the remainder 12. Find the number.

29. A number consists of three digits of which the second is 2 greater than the first, and the third, equal to the sum of the first two. If the first and third change places, the new number will be 297 more than the original number. Find the original number.

30. A can do a piece of work in 10 days, and B can do the same work in 15 days. A works alone a number of days, after which B finishes it alone in $12\frac{1}{2}$ days. How long did A work?

31. A can do a piece of work in 20 days, and B can do the same work in 30 days. They work together for a time, and then A finishes it alone in 5 days. How many days did they work together?

32. A detachment of men has a supply of bread for 30 days, allowing each man 14 ounces per day. After 15 days

200 men are withdrawn and the allowance per man is increased to 20 ounces per day, which exhausts the supply in $11\frac{2}{3}$ days. How many men were there at first?

33. A man engaged to work eight months for $200 and a suit of clothes of a certain value. At the end of six months he was discharged for bad conduct and made to forfeit one month's pay. If he received $113 and the suit of clothes, what was the value of the suit?

34. A and B engaged in trade, A with $250 and B with $300. A lost $\frac{3}{2}$ as much as B lost, and B had $\frac{3}{2}$ as much as A remaining. How much did each lose?

35. A agrees to do a certain piece of work for $500. After working 10 days alone he employs B and C, and the three together finish it in $39\frac{3}{23}$ days. If B can do half as much as A, and C $\frac{8}{5}$ as much as B, what are the daily wages of each?

36. A has $1200 a year more income than B and each saves $\frac{1}{6}$ of his income. If at the end of 5 years A has saved $\frac{7}{5}$ as much as B, what is the income of each?

37. A man spends $\frac{1}{3}$ of his money and then earns $1000. He now spends $\frac{1}{5}$ of all he has and afterwards earns $1500. If the amount which he now has will yield at 3 per cent an annual income of $165, how much had he at first?

38. A box of oranges was bought at the rate of 15 cents a dozen. Five dozen were given away and the remainder sold at the rate of 2 for 5 cents. If this gave a profit of 30 cents on the box, how many were there in the box?

39. A man gave away half a dollar more than half his money, then half a dollar more than half of what remained, and finally a fourth of a dollar more than a fourth of what still remained, and had 50 cents left. How much had he at first?

40. After giving away half a dollar less than half my money, and a third of a dollar more than a third of what remained, and then a fourth of a dollar less than a fourth of what still remained, I had $4. How much had I at first?

41. A sum of money is made up of dimes, half-dimes, and cents. There are 12 dimes, as many cents as there are dimes plus ½ of the number of half-dimes, and as many half-dimes as dimes and cents together. Find the sum of money.

42. A number consists of four digits of which the left-hand one is 4. If this digit is placed at the right of the others, the new number will be 1062 larger than the original number. Find the original number.

43. A number consists of three digits. The one in units' place is 5. If this digit is placed at the left of the others and the new number is divided by the original one, the quotient will be 4 and the remainder 12. Find the original number.

44. A company of men was drawn up in a hollow square 4 deep. Afterwards it was separated into two detachments. One was a hollow square 3 deep and with 34 more men in front than at first, and the other contained 80 men. How many men in the original company?

45. A man walks along a railway at the rate of 4 miles an hour. If a train 208 yards long and travelling 30 miles an hour overtakes him, how long will it take in passing the man?

46. One cask contains a mixture of 14 gallons of wine and 16 gallons of water; another contains a mixture of 20 gallons of wine and 10 gallons of water. How many gallons must be drawn from each to make a mixture of 5 gallons of wine and 5 gallons of water?

47. A man rowed down a stream 18 miles in 3 hours, but upon returning found that it took him 3 hours to reach a place 6 miles below his starting-point. Find the rate per hour of the rowing.

48. A man rowed down a stream 11 miles in $1\frac{1}{2}$ hours, but found that it took him $3\frac{3}{4}$ hours to return the same distance. Find the rate per hour of the stream.

49. A boatman can row 6 miles in 4 hours against a certain tide. If the tide were as swift again, it would gain upon him $1\frac{1}{2}$ miles in an hour. Find the rate per hour of the tide.

50. A hare is 56 of her own leaps ahead of a greyhound, and takes 6 leaps to the greyhound's 5; but three of the greyhound's leaps are equivalent to 5 of the hare's. How many leaps will each take before the hare is caught?

51. A hare is 50 greyhound's leaps ahead of a greyhound, and takes 8 leaps to the greyhound's 7; but 5 of the greyhound's leaps are equivalent to 8 of the hare's. How many leaps will the hare take before being caught?

52. A dinner was ordered for 20 persons at such a rate as to gain 25 per cent on the cost; but 5 of them being absent, and the others paying only the original rate, there was a loss of $1.25. What was the rate for each person?

53. A regiment can be drawn up in two hollow squares, one three deep and the other five deep. If the front line in the first contains 6 men more than the front line in the second, and both squares contain the same number of men, how many men in the regiment?

54. A man has $25000. He invests a portion poorly and gets no income from it. One-fifth of the remainder pays him 4 per cent; three-fourths of what still remains, 6 per cent; and the other fourth, 5 per cent. If his income is $1080 per year, what is the amount of the poor investment?

55. At what time between 5 and 6 will the minute hand of a clock be 8 minute spaces in advance of the hour hand?

56. At what time between 11 and 12 will the hands of a clock be at right angles to each other?

57. The number of candidates who passed a certain examination was three times the number rejected; but if there had been 14 more candidates and 4 less rejected, the number who passed would have been four times the number rejected. How many candidates were there?

58. A makes 5 perfect recitations out of 12, and B 7 out of 15. A recites $\frac{4}{5}$ as many times as B, and they both together make 48 perfect recitations. How many times did each recite?

59. The sum of three numbers is 85. If the third is divided by the first, the quotient will be 2 with a remainder of 2; but if the third is divided by the second, the quotient will be 1 with a remainder of 19. What are the numbers?

60. A certain piece of work can be done by 5 men in 6 days, or by 8 women in 12 days, or by 15 boys in 12 days. How many days will it take 1 man, 1 woman, and 1 boy working together?

61. The distance from A to B is 32 miles. A courier sets out from A and reaches B 12 minutes after a second courier, who left A when the first courier had proceeded 11 miles upon his journey. If the second courier can make the journey in 4 hours, at what distance from B did he pass the first courier?

62. A and B start at the same time to walk from C to D, at the rate of $3\frac{1}{3}$ and 4 miles per hour respectively. After B has gone $\frac{2}{3}$ of the distance he rests $\frac{3}{5}$ of an hour, which allows A to overtake him. Find the distance from C to D.

63. A certain sum of money is made up of twenty-five-cent pieces and $\frac{3}{5}$ as many dimes. Another sum, 20 cents less, is made up of as many twenty-five-cent pieces as there are dimes, and $\frac{3}{2}$ as many dimes as there are twenty-five-cent pieces in the first sum. What is the amount of the first mentioned sum?

64. A man bequeathed his property as follows: To his wife $1200 and $\frac{1}{5}$ of what remained; to his eldest son $\frac{1}{2}$ as much and $\frac{1}{5}$ of what then remained; and to his second son $\frac{1}{2}$ as much as to the eldest son, and $\frac{1}{2}$ of what still remained. If $3776 yet remained, what was the whole estate?

65. A number of marbles was divided among some boys. The first took 5 and $\frac{1}{6}$ of what remained; the second took 10 and $\frac{1}{6}$ of what remained; the third took 15 and $\frac{1}{6}$ of what still remained, and in like manner the others until all were taken, when it was found that each boy had the same number. How many marbles were there?

66. There are two roads from A to B. Two men whose rates of walking are respectively 4 and 5 miles per hour, start at the same time from A to B by different roads, and return to A each by the road the other travelled to B. They meet in B, and one reaches A 27 minutes before the other. Find the lengths of the two roads from A to B.

67. A and B run a race; at the end of 5 minutes when A has run 900 yards and beaten B by 75 yards he falls, and for the rest of the race makes 20 yards less per minute than at first, yet comes in only half a minute behind B. What was B's time?

SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

DEFINITIONS.

114. **Independent Equations** are such as cannot be reduced to the same form.

115. **Simultaneous Equations** are independent equations which are satisfied by the same values of their unknown quantities.

116. **Elimination** is the process of combining a given set of simultaneous equations so as to produce one or more new equations containing less unknown quantities.

117. NUMERICAL SIMULTANEOUS EQUATIONS.

Solve the following equations:

1. $\begin{cases} 2x - 3y = -3. \\ 3x - 2y = 8. \end{cases}$

2. $\begin{cases} 5x + 3y = 68. \\ 3x - 2y = 18. \end{cases}$

3. $\begin{cases} 16x - 16y - 1 = \frac{4}{3}. \\ 2x + y + 6 = \frac{22}{3}. \end{cases}$

4. $\begin{cases} 2y - 3x = 6. \\ 2x - 3y = -24. \end{cases}$

5. $\begin{cases} x - \dfrac{5y+2}{2} = -4. \\ y - \dfrac{2x-3}{7} = 3. \end{cases}$

6. $\begin{cases} \dfrac{8x-3}{4} + \dfrac{y-5}{3} = \dfrac{7}{6}. \\ x - \dfrac{y-7}{5} = \dfrac{5}{8}. \end{cases}$

7. $\begin{cases} \dfrac{\frac{3x}{2} - \frac{2y}{3}}{\frac{11}{2}} - \dfrac{4y-3x}{7} = -\dfrac{4}{3}. \\ y + x = 7(y - x). \end{cases}$

8. $\begin{cases} \dfrac{\frac{2x}{3} - \frac{5y}{12}}{\frac{7}{4}} - \dfrac{\frac{3x}{2} - \frac{y}{3}}{\frac{23}{2}} = 2. \\ 2x - y = 4(2y - x). \end{cases}$

9. $\begin{cases} \dfrac{2x-3}{4} - \dfrac{y-8}{5} = \dfrac{y+3}{4}. \\ \dfrac{x+y}{3} + \dfrac{4y+1}{11} = 8. \end{cases}$

10. $\begin{cases} \dfrac{x-y}{2} - \dfrac{x-3y}{5} = y - 3. \\ \frac{3}{4}(x - y) + \frac{5}{6}(x + y) = 18. \end{cases}$

11. $\begin{cases} \dfrac{9}{2x+3y+6} = \dfrac{17}{9x+2y+3}. \\ \dfrac{5x-7y+2}{9} = -\dfrac{2x+5y-17}{3}. \end{cases}$

12. $\begin{cases} \dfrac{12}{x} + \dfrac{8}{y} = 8. \\ \dfrac{27}{x} - \dfrac{12}{y} = 3. \end{cases}$

13. $\begin{cases} \dfrac{15}{x} - \dfrac{10}{y} = 3. \\ \dfrac{9}{x} + \dfrac{16}{y} = 6\tfrac{1}{5}. \end{cases}$

14. $\begin{cases} \dfrac{5}{x} + \dfrac{3}{y} = \dfrac{7}{20}. \\ 15x + 10y = xy. \end{cases}$

15. $\begin{cases} 2y - x = 4xy. \\ \dfrac{4}{y} - \dfrac{3}{x} = 9. \end{cases}$

16. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = 1. \\ 2x + y = 7xy. \end{cases}$

17. $\begin{cases} 2x + 3y = 5. \\ \dfrac{5}{x} + \dfrac{7}{y} = \dfrac{12}{xy}. \end{cases}$

18. $\begin{cases} \dfrac{10}{3x} - \dfrac{5}{2y} = \dfrac{5}{6}. \\ \dfrac{10}{x} - \dfrac{5}{y} = \dfrac{10}{3}. \end{cases}$

19. $\begin{cases} \dfrac{2}{3x} + \dfrac{3}{y} = \dfrac{16}{45}. \\ \dfrac{3}{2x} - \dfrac{1}{2y} = \dfrac{3}{40}. \end{cases}$

118. LITERAL SIMULTANEOUS EQUATIONS.

Solve the following equations:

1. $\begin{cases} ax + by = c. \\ bx - ay = d. \end{cases}$

2. $\begin{cases} x + y = a + b. \\ bx + ay = 2ab. \end{cases}$

3. $\begin{cases} \dfrac{x}{b} + \dfrac{y}{c} = 1. \\ \dfrac{ax}{c} - \dfrac{by}{a} = 0. \end{cases}$

4. $\begin{cases} \dfrac{x}{c} + \dfrac{y}{d} = 1. \\ x - y = c - d. \end{cases}$

5. $\begin{cases} \dfrac{ax}{c} + \dfrac{by}{d} = a + b. \\ \dfrac{x}{3c} + \dfrac{y}{6d} = \dfrac{1}{2}. \end{cases}$

6. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{a}. \\ \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{b}. \end{cases}$

7. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = c. \\ \dfrac{b}{x} + \dfrac{a}{y} = d. \end{cases}$

8. $\begin{cases} x + y = a + b. \\ \dfrac{x + a}{y + b} = \dfrac{b}{a}. \end{cases}$

9. $$\begin{cases} x\,(a+b)+y\,(a-b)=2. \\ ax+by=\dfrac{a^2+b^2}{a^2-b^2}. \end{cases}$$

10. $$\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=2. \\ \dfrac{x}{b}+\dfrac{y}{a}=\dfrac{a^2+b^2}{ab}. \end{cases}$$

11. $$\begin{cases} \dfrac{x+y}{a}+\dfrac{x-y}{b}=1. \\ \dfrac{x-y}{a}-\dfrac{x+y}{b}=1. \end{cases}$$

12. $$\begin{cases} \dfrac{x+1}{y+1}=\dfrac{m+n+1}{m-n+1}. \\ x\,(m+n)+y\,(m+n+1)=2m^2+2mn+m-n. \end{cases}$$

13. $$\begin{cases} \dfrac{x}{a+b}+\dfrac{y}{a-b}=2a. \\ \dfrac{x-y}{4ab}=1. \end{cases}$$

14. $$\begin{cases} \dfrac{x}{2a}+\dfrac{y}{3b}=\dfrac{5}{6}. \\ \dfrac{x}{5a}-\dfrac{y}{2b}=\dfrac{8}{5}. \end{cases}$$

15. $$\begin{cases} \dfrac{x}{2a}+\dfrac{y}{3b}=\dfrac{5}{6}. \\ \dfrac{x}{a}+\dfrac{y}{b}=2. \end{cases}$$

16. $$\begin{cases} (a-b)\,x+(a+b)\,y=2\,a^2-2\,b^2. \\ (a+b)\,x+(a-b)\,y=2\,(a^2+b^2). \end{cases}$$

119. PROBLEMS PRODUCING SIMPLE EQUATIONS OF TWO UNKNOWN QUANTITIES.

1. If A gives B \$100, B will have six times as much as A; but if B gives A \$100, A will have two and a half times as much as B. How much has each?

2. Eight years ago A was five times as old as B, and in two years he will be three times as old. Find their present ages.

3. Thirty bushels of wheat and 25 bushels of corn cost \$49, and 50 bushels of wheat and 10 bushels of corn cost \$50. Find the price of each per bushel.

4. What fraction equals $\frac{2}{3}$ when 3 is added to both numerator and denominator, and equals $\frac{5}{7}$ when 5 is added in the same way?

5. There are two fractions, of which the first is larger, whose difference is $\frac{1}{3}$ when the denominators are 10 and 12, but whose difference is $\frac{16}{45}$ when the denominators are 9 and 10. Find the fractions.

6. If the numerator of a certain fraction is multiplied by 4, and 5 is added to the denominator, the value of the resulting fraction will be 1; but if the denominator is multiplied by 4, and 5 is added to the numerator, the value of the resulting fraction will be $\frac{2}{7}$. Find the fraction.

7. A and B can do a piece of work together in 30 days. After they have both worked 12 days B is called away and A finishes it alone in 24 days. How many days would it take each alone?

8. A and B can do a piece of work together in 12 days. A works 3 days alone, they both work together 5 days, and B finishes it alone in 13 days. How many days would it take each alone?

9. Two sums of money are loaned, the first at 5 per cent and the second at 4 per cent, and both together yield an annual income of $68. If both sums were invested in 6 per cent bonds costing 125, the income would be $72. Find the sums of money.

10. A man invests $5000 in 6 per cent bonds at 94, and in 7 per cent bonds at 104. If the total annual income is $330, what amount is invested at each rate?

11. A has a sum of money invested at a certain rate; B has $500 more invested at a rate 1 per cent higher and receives $85 per year more income than A; C has $1000 more than A invested at a rate 1 per cent lower than A, but receives the same income as A. Find the capital and rate of each.

12. A and B together invest $2000. A's money was in trade 12 months and he received $180 profit; B's money was in trade 7 months and he received $315 profit. How much did each invest?

13. A number is composed of two digits whose sum is 7. If their order is inverted, the new number will be 2 more than twice the original number. Find the original number.

14. The first of the two digits of a number is $\frac{1}{3}$ of the second. If their order is inverted and the new number is divided by the original number, the quotient will be 2 and the remainder 10. Find the original number.

15. The two digits of a number change places if 9 is subtracted from the number; the same result is obtained by adding 7 to $\frac{1}{2}$ of the number. Find the number.

16. A number with two digits divided by the sum of its digits plus 2 gives a quotient 6 and a remainder 6. If the order of the digits is inverted and the resulting number is divided by the sum of its digits plus 3, the quotient will be 2 and the remainder 3. Find the number.

17. A number consists of three digits, of which the third equals the sum of the first two. If the number is divided by the sum of its digits the quotient will be 23 and the remainder 5, and if the unit figure is put into hundredths' place, the others remaining in the original order, the number will be increased 288. Find the number.

18. Each of two persons owes $500. The first says to the second, give me $\frac{1}{3}$ of your money and I can pay my debt. The second says to the first, I can pay your debt and have a sum equal to $\frac{1}{3}$ of your money remaining. How much has each person?

19. A and B race from C to D, the distance being 100 miles. When B has a start of 16 miles he is beaten by one hour, but when given 6 hours' start wins by 5 miles. Find the rate per hour of each.

20. A and B run a race of 300 yards. When A has a start of 3 seconds he is beaten by 21 yards; when he has a start of 10 yards he is beaten by $6\frac{2}{3}$ seconds. Find the rate per second of each.

21. A person rows a certain distance down a stream, which runs at the rate of 3 miles an hour, in $3\frac{2}{3}$ hours. In returning it takes him 10 hours to reach a place three miles below his starting-point. Find the distance down the stream and the rate of rowing.

22. A man can row 24 miles down a stream and back to the starting-point in 15 hours. He finds he can row as far in 1 hour going down as in 4 hours going up the stream. Find the rate of the current and of the rowing.

23. A boat's crew row 15 miles with the tide in $\frac{3}{4}$ of an hour; when the tide is twice as swift they row 24 miles against it in 3 hours. Find the rate of rowing and of the swiftest tide.

24. A rectangle is of the same area as another which is 4 yards longer and 1 yard narrower; it is also of the same area as a third, which is 4 yards shorter and 2 yards wider. What is its area?

25. A lost $\frac{4}{5}$ of his money and borrowed of B such a sum that each then had $\frac{2}{3}$ as much as at first, and A $8 less than B. How much had each at first?

26. A and B engage in play with equal sums. After a time A is found to have $\frac{1}{2}$ as much as B, but if he had had $$16\frac{2}{3}$ less, he would have had $\frac{1}{3}$ as much as B. Find what each had at first, and how much A lost.

27. A cask contains a mixture of 12 gallons of wine and 4 gallons of water; another cask contains a mixture of 8 gallons of wine and 12 of water. How many gallons must be drawn from each to make a mixture of 7 gallons of wine and 7 of water?

28. A railway train after running 5 hours meets with an accident which delays it 30 minutes, after which it proceeds at $\frac{3}{4}$ of its usual speed, and reaches its destination 3 hours late. If the accident had happened 40 miles nearer the starting-point, it would have been $3\frac{1}{3}$ hours late. Find the regular speed and the length of the line.

29. An income of $145 is derived from a sum of money invested partly in $5\frac{1}{2}$ per cent and partly in 4 per cent bonds. If the $5\frac{1}{2}$ per cent bonds are sold at 110, and the 4 per cent bonds at 120, the amount realized is $3140. How much stock of each kind was there?

30. A boat goes up stream 30 miles and down stream 44 miles in 10 hours. Again it goes up stream 40 miles and down stream 55 miles in 13 hours. Find the rates per hour of the stream and of the boat.

SIMPLE EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

120. NUMERICAL AND LITERAL EQUATIONS.

Solve the following equations:

1. $$\begin{cases} 2x + 3y + z = 86. \\ 5x - y - z = 26. \\ 4x - y + 3z = 94. \end{cases}$$

2. $$\begin{cases} 2x + 3y + 5z = 3. \\ 4x - 6y + z = \frac{1}{5}. \\ x - y - z = -\frac{1}{30}. \end{cases}$$

3. $$\begin{cases} 4x + 2y = 50. \\ y + z = 6. \\ x + z = 11. \end{cases}$$

4. $$\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{4} = 8. \\ \dfrac{x}{4} + \dfrac{y}{9} - \dfrac{z}{6} = 0. \\ \dfrac{x}{3} + \dfrac{y}{4} + \dfrac{z}{5} = \dfrac{359}{60}. \end{cases}$$

5. $$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 8. \\ \dfrac{1}{y} + \dfrac{1}{z} = 15. \\ \dfrac{1}{x} + \dfrac{1}{z} = 13. \end{cases}$$

6. $$\begin{cases} \dfrac{2}{x}+\dfrac{3}{y}+\dfrac{1}{z}=\dfrac{41}{40}. \\ \dfrac{3}{x}-\dfrac{2}{y}+\dfrac{3}{z}=\dfrac{77}{120}. \\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{59}{120}. \end{cases}$$

7. $$\begin{cases} \dfrac{1}{2x}+\dfrac{1}{2y}+\dfrac{1}{2z}=\dfrac{13}{24}. \\ \dfrac{1}{3x}-\dfrac{1}{5y}+\dfrac{1}{z}=\dfrac{7}{20}. \\ \dfrac{1}{4x}-\dfrac{1}{y}-\dfrac{1}{3z}=-\dfrac{7}{24}. \end{cases}$$

8. $$\begin{cases} \dfrac{x}{3+y}=\dfrac{3}{7}. \\ \dfrac{y}{4+z}=\dfrac{4}{9}. \\ \dfrac{z}{x+5}=\dfrac{5}{8}. \end{cases}$$

9. $$\begin{cases} \dfrac{xy}{x+y}=70. \\ \dfrac{xz}{x+z}=84. \\ \dfrac{yz}{y+z}=140. \end{cases}$$

10. $$\begin{cases} x+y+z+u=36. \\ 4x+y-z=22. \\ 4z-u=28. \\ 2u+y=32. \end{cases}$$

11. $$\begin{cases} x+y+z=6. \\ y+z+u=9. \\ z+u+x=8. \\ u+x+y=7. \end{cases}$$

12. $$\begin{cases} 2x+y+z+u=63. \\ x+y+2z+2u=61. \\ 3x+4y+3z+4u=142. \\ x-y+z-3u=1. \end{cases}$$

13. $$\begin{cases} 7x-3y=1. \\ 11z-7u=1. \\ 5z+6y=134. \\ 2x+u=33. \end{cases}$$

14. $$\begin{cases} 2x + 3z - u = 25. \\ 3z + y + 2v = 33. \\ 6y - x + u = 14. \\ 10y + u + 3v = 43. \\ 2z + u = 22. \end{cases}$$

15. $$\begin{cases} ay + bx = c. \\ cx + az = b. \\ bz + cy = a. \end{cases}$$

16. $$\begin{cases} (b - c)x + (c - a)y + (a - b)z = 0 \\ ax + (a - c)y - cz = 0. \\ x + y + z = a + b + c. \end{cases}$$

17. $$\begin{cases} x + y + z = 0. \\ ax + by + cz = 0. \\ bcx + acy + abz + (b - c)(c - a)(a - b) = 0. \end{cases}$$

121. PROBLEMS PRODUCING SIMPLE EQUATIONS OF THREE OR MORE UNKNOWN QUANTITIES.

1. Three boys have together 80 marbles. If A gives $\frac{1}{2}$ of his to B and C in equal shares, A and C will have the same number, and B as many as both together. How many had each at first?

2. Three men were met by robbers who took from A a sum equal to $\frac{1}{3}$ of B's and $\frac{1}{10}$ of C's money, from B a sum equal to $\frac{1}{2}$ of A's and $\frac{1}{5}$ of C's money, and from C a sum equal to $\frac{1}{4}$ of A's and B's money together. It was then found that A had $\frac{1}{2}$ as much as B and $\frac{2}{15}$ as much as C, and B $275 less than C. How much had each at first?

3. There is a number consisting of three digits. The sum of the first and second digits exceeds the third by 4. The sum of the three digits is 10; and, when the order of the digits is inverted, the original number is increased by 99. Find the number.

4. A, B, C, and D are situated in the given order upon the same road. The distance between A and B is 6 miles greater than the distance between C and D; the distance between A and C is $\frac{1}{16}$ of a mile less than $\frac{3}{2}$ as great as that between B and D; and the point half way between A and D is between B and C, and $\frac{1}{2}$ a mile from B. Find the distances AB, BC, and CD.

5. A, B, and C can do a piece of work in 20 days; A and B, in 30 days; and B and C, in 40 days. How long would it take each to do the work alone?

6. A, B, and C can do a piece of work in 15 days. A and B together can do $\frac{4}{3}$ as much as C, and C twice as much as A. How long would it take each to do the work alone?

7. A school is divided into three classes. The first and second contain 50 pupils more than half the whole number; the second and third 30 more than half the whole; and the first and third 10 more than half the whole. Find the number in each class.

8. A number consists of three digits of which the sum of the first two equals the third. If the number is divided by the sum of its digits, the quotient will be 23 and the remainder 5; and, if 297 is added to the number, the order of the digits will be inverted. Find the number.

9. A number consists of four digits whose sum is 10. The difference between the first and second equals the difference between the third and fourth; and, if the first and fourth change places, the new number exceeds the old by 2997; but, if the second and third change places, the new number exceeds the old by 90. Find the number.

10. A cistern has three pipes, A, B, and C. A and B can fill it in 20 minutes; A and C, in $16\frac{2}{3}$ minutes; and B and C, in $14\frac{2}{7}$ minutes. Find the time required for each alone to fill it.

11. A cistern has three pipes, A, B, and C. If A and B run in while C runs out, it will be filled in $28\frac{4}{7}$ minutes. If B and C run in while A runs out, it will be filled in 40 minutes. If A and C run in while B runs out, it will be filled in $66\frac{2}{3}$ minutes. In what time would each fill it alone, the others not running?

12. In a 10-mile race A can beat B by 2 miles, and C by 4 miles. By how many miles can B beat C?

13. $1000 is divided among A, B, C, and D. B gets half as much as A; the excess of C's share over D's share is equal to one-third of A's share, and if B's share were increased by $100 he would have as much as C and D have between them. Find how much each gets.

XII. THE BINOMIAL THEOREM.

122. EXPANSION OF BINOMIALS.

POSITIVE INTEGRAL EXPONENTS.

Expand the following:

1. $(a + b)^4$.
2. $(a - b)^5$.
3. $(x - y)^6$.
4. $(a - 1)^4$.
5. $(a^2 + b^2)^4$.
6. $(a^3 + b^3)^5$.
7. $(a^3 - b^2)^3$.
8. $(a + b^2)^5$.
9. $(2a - 3b)^5$.
10. $(3a - 2b)^6$.
11. $(4a^2 + 5b^2)^4$.
12. $(1 - 3x)^5$.
13. $(4x^2 - 1)^6$.
14. $(2a^2 + 3b^2)^4$.
15. $(4a^2 + 2ab)^4$.
16. $(x - 5x^2)^5$.
17. $(2ab - b^2)^4$.
18. $(2x^2 - 5y)^4$.
19. $(2a^2 - 5b)^5$.
20. $\left(\frac{a}{2} - \frac{b}{3}\right)^4$.

21. $\left(2a - \frac{b}{2}\right)^6.$

22. $\left(\frac{2a}{3} + \frac{b}{2}\right)^4.$

23. $\left(\frac{3x}{2} - 1\right)^8.$

24. $\left(2ax - \frac{b}{3}\right)^5.$

NEGATIVE AND FRACTIONAL EXPONENTS.

Expand those which do not terminate to four terms:

25. $(a^{-2} + b^{-2})^8.$

26. $(a^{-1} + b^2)^4.$

27. $(a^{\frac{1}{2}} - b^{\frac{1}{2}})^7.$

28. $(a^{\frac{1}{2}} + b^{-\frac{1}{2}})^5.$

29. $(a^{-\frac{1}{2}} - b^{-\frac{1}{2}})^8.$

30. $(4a^{-2} - b^{-1})^5.$

31. $(a^2 + b^2)^{-1}.$

32. $(a^2 - b^2)^{-2}.$

33. $(a^{-4} - b^{-4})^{\frac{1}{2}}.$

34. $(a^{-1} + b^{-1})^{-\frac{1}{2}}.$

35. $(8a^2 - b^2)^{\frac{1}{3}}.$

36. $\left(\frac{2a}{3} - \frac{b}{2}\right)^{-2}.$

37. $(\frac{3}{2}a^{-\frac{1}{2}} + \frac{1}{2}b^{-\frac{1}{2}})^{-8}.$

38. $\frac{1}{(a-2b)^2}.$

39. $\frac{1}{(16a-b)^{\frac{1}{4}}}.$

40. $\frac{2}{(9a+2b)^{-\frac{1}{2}}}.$

41. $(1+2x)^{-\frac{1}{2}}.$

42. $(9-6x)^{-\frac{3}{2}}.$

43. $(4a-8b)^{-\frac{1}{2}}.$

44. $(8+12x)^{\frac{2}{3}}.$

45. $(x+12)^{\frac{1}{2}}.$

46. $(1-3x)^{-\frac{1}{3}}.$

47. $(128a^{-7} - 3b)^{-\frac{1}{7}}.$

48. $(1728a^{-1} - 12b^{-1})^{-\frac{1}{3}}.$

TO EXTRACT ROOTS BY THE BINOMIAL THEOREM.

Find approximate roots as indicated:

49. $\sqrt{98}.$

50. $\sqrt[5]{31}.$

51. $\sqrt[4]{620}.$

52. $\sqrt[3]{122}.$

53. $(128)^{-\frac{1}{3}}.$

54. $(998)^{\frac{1}{3}}.$

55. $(2400)^{\frac{1}{4}}.$

56. $(630)^{-\frac{3}{4}}.$

123. TO FIND ANY PARTICULAR TERM OF AN EXPANSION.

POSITIVE INTEGRAL EXPONENTS.

Find the

1. Fourth term of $(x - y)^{12}$.
2. Fifth term of $(x + y)^{6}$.
3. Fifth term of $(a + b)^{16}$.
4. Sixth term of $(a^2 - b^2)^{10}$.
5. Eighth term of $(m - 1)^{12}$.
6. Tenth term of $(x - y)^{20}$.
7. Sixth term of $(a^3 + b^3)^{11}$.
8. Fifth term of $(2a + 3b)^{8}$.
9. Sixth term of $(3x - y)^{7}$.
10. Fourth term of $(2x - \frac{1}{2}y)^{12}$.
11. Fourth term of $\left(\frac{x^2}{2} - \frac{y^2}{3}\right)^{15}$.
12. Seventh term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^{9}$.
13. Middle term of $\left(\frac{2x}{3} - \frac{3a}{2}\right)^{6}$.
14. Term containing x^{16} in $\left(x^2 - \frac{a}{2x}\right)^{14}$.
15. Term containing a^8 in $(1 - 3a^2)^{8}$.
16. Term not containing x in $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^{9}$.
17. Term not containing x in $\left(x - \frac{1}{x^2}\right)^{12}$.
18. Fifteenth term of $(a^2 - 2b^2)^{18}$.

NEGATIVE AND FRACTIONAL EXPONENTS.

Find the

19. Fifth term of $(a^{-2} - b^{-2})^{-2}$.

20. Fourth term of $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^{\frac{1}{3}}$.

21. Fourth term of $(a - b^{\frac{2}{3}})^{-\frac{1}{2}}$.

22. Sixth term of $(x - y)^{-\frac{2}{3}}$.

23. Fifth term of $(4x - y^{\frac{1}{2}})^{\frac{1}{2}}$.

24. Sixth term of $(9x^{-1} - y^{-1})^{-\frac{1}{2}}$.

25. Fourth term of $\dfrac{1}{(x - 2y)^{\frac{1}{2}}}$.

26. Seventh term of $(2x - 3y)^{-3}$.

27. Sixth term of $(27x^3 - 8y^3)^{\frac{1}{3}}$

28. Fifth term of $(8x - y)^{-\frac{2}{3}}$.

29. Fifth term of $(8a^{-2} - 3b^{-1})^{\frac{2}{3}}$.

30. Tenth term of $(16x^{-2} + 3y)^{-1}$.

XIII. EXTRACTION OF ROOTS.

124. SQUARE ROOT.

ALGEBRAIC EXPRESSIONS.

1. Write the formula for extracting the square root, and show how it is obtained.

Extract the square roots of the following:

2. $4 - 12x - 7x^2 + 24x^3 + 16x^4$.

3. $x^4 + 2x^3y - x^2y^2 - 2xy^3 + y^4$.

4. $1 - 6x + 15x^2 - 18x^3 + 9x^4.$

5. $x^4 + 6x^3 + 11x^2 + 6x + 1.$

6. $9x^4 - 12x^3 + 10x^2 - 4x + 1.$

7. $16x^4 + 16x^3 + 12x^2 + 4x + 1.$

8. $4x^6 + 4x^5 - 3x^4 - 6x^3 - x^2 + 2x + 1.$

9. $a^6 - 2a^5 - a^4 + 3a^2 + 2a + 1.$

10. $a^6 - a^5 - \frac{7a^4}{4} - a^3 + 2a^2 + 2a + 1.$

11. $9x^8 - 12x^7 + 22x^6 - 18x^5 + 19x^4 - 10x^3 + 7x^2 - 2x + 1.$

12. $\frac{x^4}{4} + 4x^2 + \frac{ax^2}{3} + \frac{a^2}{9} - 2x^3 - \frac{4ax}{3}.$

13. $\frac{x^4}{4} + \frac{x^3}{y} + \frac{x^2}{y^2} - xy - 2 + \frac{y^2}{x^2}.$

14. $\frac{4x^2}{9y^2} - \frac{x}{z} - \frac{16x^2}{15yz} + \frac{9y^2}{16z^2} + \frac{6xy}{5z^2} + \frac{16x^2}{25z^2}.$

15. $x^{\frac{5}{3}} - 4x^{\frac{4}{3}} + 2x + 4x^{\frac{2}{3}} + x^{\frac{1}{3}}.$

16. $x^{\frac{2}{3}} - 4x^{\frac{5}{6}} + 4x + 2x^{\frac{7}{6}} - 4x^{\frac{4}{3}} + x^{\frac{5}{3}}.$

17. $x^{\frac{8}{5}} - 2a^{-\frac{3}{5}}x^{\frac{11}{5}} + 2a^{\frac{4}{5}}x^{\frac{4}{5}} + a^{-\frac{6}{5}}x^{\frac{14}{5}} - 2a^{\frac{1}{5}}x^{\frac{7}{5}} + a^{\frac{8}{5}}.$

18. $1 + x + 2x^2.$ 19. $4 + 2x + x^2.$ 20. $x^2 - 1.$

NUMBERS.

Extract the square roots of the following, using the algebraic formula:

21. 7918596.
22. 1863821584.
23. 291.0436.
24. 25623844.
25. 2250300.01.
26. 56875730.56.
27. 216.15174441.
28. 494210406001.
29. 129.
30. .00852.
31. 6.21.
32. 12.

125. CUBE ROOT.

ALGEBRAIC EXPRESSIONS.

1. Write the formula for extracting the cube root, and show how it is obtained.

Extract the cube roots of the following:

2. $a^3 + 6a^2b + 12ab^2 + 8b^3$.
3. $8a^3 + 36a^2b + 54ab^2 + 27b^3$.
4. $8x^6 - 36x^4 + 54x^2 - 27$.
5. $8x^3 - 60x^2 + 150x - 125$.
6. $27x^6 + 27x^5 + 36x^4 + 19x^3 + 12x^2 + 3x + 1$.
7. $125x^6 - 225x^5 - 15x^4 + 153x^3 + 6x^2 - 36x - 8$.
8. $125a^9 + 75a^8 + 15a^7 - 74a^6 - 30a^5 - 3a^4 + 15a^3 + 3a^2 - 1$.
9. $x^9 - 3x^8 + 8x^6 - 6x^5 - 6x^4 + 8x^3 - 3x + 1$.
10. $8x^9 + 36x^8 + 30x^7 - 33x^6 + 6x^5 + 39x^4 - 38x^3 + 21x^2 - 6x + 1$.
11. $171x^4 - 27x^5 - 109x^3 - 108x + 342x^2 + 216 + 27x^6$.
12. $8x^6 + 48bx^5 + 60b^2x^4 - 80b^3x^3 - 90b^4x^2 + 108b^5x - 27b^6$.
13. $x^3 - 9x + \frac{27}{x} - \frac{27}{x^3}$.
14. $\frac{x^3}{y^3} + \frac{9x^2}{y^2} + \frac{24x}{y} + 9 - \frac{24y}{x} + \frac{9y^2}{x^2} - \frac{y^3}{x^3}$.
15. $\frac{x^3}{a^3} - \frac{9x^2}{a^2} + \frac{33x}{a} - 63 + \frac{66a}{x} - \frac{36a^2}{x^2} + \frac{8a^3}{x^3}$.
16. $8x^3 - 12x^2 + 30x - 25 + 30x^{-1} - 12x^{-2} + 8x^{-3}$.
17. $27x^3 - 54x^{\frac{5}{2}}y^{\frac{1}{8}} + 63x^2y^{\frac{1}{4}} - 44x^{\frac{3}{2}}y^{\frac{3}{8}} + 21xy^{\frac{1}{2}} - 6x^{\frac{1}{2}}y^{\frac{5}{8}} + y^{\frac{3}{4}}$.

18. $x^{\frac{3}{4}} + 6x^{\frac{5}{8}}y^{-1} + 3x^{\frac{1}{2}}y^{-2} - 28x^{\frac{3}{8}}y^{-3} - 9x^{\frac{1}{4}}y^{-4}$
$+ 54x^{\frac{1}{8}}y^{-5} - 27y^{-6}$.

19. $1 + x - x^2$ to four terms.

20. $1 + 2x + 2x^2$ to four terms.

NUMBERS.

Extract the cube roots of the following, using the algebraic formula:

21. 13824.
22. 46656.
23. 19683.
24. 42875.
25. 2628072.
26. 60236.288.
27. 54010152.
28. 191.102976.
29. 2.25.
30. 42.6.
31. .0823.
32. 144182818617453.

126. FOURTH ROOT.

Find the fourth roots of:

1. $16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$.

2. $81x^8 - 108x^6y + 54x^4y^2 - 12x^2y^3 + y^4$.

127. FIFTH ROOT.

Find the fifth roots of:

1. $x^5 + 20x^4 + 160x^3 + 640x^2 + 1280x + 1024$.

2. $32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243$.

128. SIXTH ROOT.

Find the sixth roots of:

1. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

2. $64a^6 + 576a^5b + 2160a^4b^2 + 4320a^3b^3 + 4860a^2b^4$
$+ 2916ab^5 + 729b^6$.

XIV. EXPONENTS.

MENTAL EXERCISES.

129. Find the values of the following:

1. $25^{\frac{1}{2}}$.
2. $64^{\frac{1}{6}}$.
3. $32^{-\frac{1}{5}}$.
4. $(-8)^{-\frac{2}{3}}$.
5. $256^{\frac{3}{4}}$.
6. $216^{-\frac{2}{3}}$.
7. $25^{-\frac{1}{2}}$.
8. $(-8)^{\frac{5}{3}}$.
9. $81^{\frac{3}{4}}$.
10. $49^{-\frac{1}{2}}$.
11. $(-512)^{\frac{2}{9}}$.
12. $-32^{\frac{3}{5}}$.
13. $125^{-\frac{2}{3}}$.
14. $(-1728)^{-\frac{1}{3}}$.
15. $25^{-\frac{3}{2}}$.
16. $(\frac{25}{16})^{-\frac{1}{2}}$.
17. $(\frac{1}{32})^{-\frac{2}{5}}$.
18. $(\frac{125}{27})^{\frac{2}{3}}$.
19. $(\frac{1}{4})^{\frac{3}{2}}$.
20. $(\frac{16}{49})^{-\frac{3}{2}}$.
21. $(5\frac{1}{16})^{\frac{3}{4}}$.
22. $(15\frac{5}{8})^{-\frac{2}{3}}$.
23. $(\frac{25}{49})^{-\frac{3}{2}}$.
24. $(\frac{32}{243})^{-\frac{1}{5}}$.

Show that:

25. $16^2 \times 12^3 = 2^{14} \times 3^3$.

26. $12^2 \times 12^3 \times 12^4 = 2^{18} \times 3^9$.

27. $10^2 \times 100^3 \times 50^4 = 5^{16} \times 2^{12}$.

28. $25^2 \times 100^{-\frac{1}{2}} \times 50^3 = 5^9 \times 2^2$.

29. $16^3 \times 20^2 \times 32^{-5} = 2^{-9} \times 5^2$.

30. $[(16)^{-\frac{3}{4}} - (81)^{-\frac{3}{4}}] \div [(16)^{-\frac{1}{4}} - (81)^{-\frac{1}{4}}] = \frac{19}{36}$.

130. DEFINITION. — A **Pure Equation** is one which contains only one power of the unknown quantity.

Solve the following equations:

1. $x^{\frac{1}{4}} = 3$.
2. $x^{\frac{2}{3}} = 16$.
3. $x^{\frac{3}{5}} = -8$.
4. $x^{-\frac{3}{4}} = -27$.
5. $x^{-\frac{2}{3}} = 4$.
6. $x^{-1} = 2$.
7. $x^{\frac{5}{3}} = -32$.
8. $x^{-\frac{1}{2}} = 5$.
9. $x^{\frac{3}{2}} = 8$.

GENERAL EXERCISES.

131. Simplify and express with positive exponents:

1. $(16\,a^{-4} \div 9\,a^{-2})^{-\frac{1}{2}}$.

2. $(x^{\frac{1}{2}}y^{\frac{3}{2}})^{\frac{1}{3}} \div (x^{-\frac{1}{3}}y)^{\frac{1}{2}}$.

3. $(a^9b^{12})^{\frac{1}{3}} + [(a^2b^8)^{\frac{1}{4}}]^2 + [(a^8b^7)^{\frac{1}{2}}]^{2\frac{1}{1}}$.

4. $\{a^{\frac{1}{2}}b^{-\frac{1}{4}}[a^{\frac{1}{3}}b^{\frac{1}{2}}(b^{\frac{2}{3}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^8$.

5. $\left(\dfrac{x^{m+n}}{x^n}\right)^m \div \left(\dfrac{x^n}{x^{m+n}}\right)^{n-m}$.

6. $(2^{n+4} - 2 \times 2^n) \div (4 \times 2^{n+2})$.

7. $\left[\left(\dfrac{ay}{x^8}\right)^{\frac{1}{2}} \times \left(\dfrac{bx^2}{y^4}\right)^{\frac{1}{3}} \times \left(\dfrac{y^8}{a^{\frac{5}{2}}b^{\frac{5}{2}}}\right)^{\frac{1}{6}}\right]^{\frac{6}{7}}$.

8. $\dfrac{a-b}{a^{\frac{1}{3}} - b^{\frac{1}{3}}} - \dfrac{a+b}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}$.

9. $\dfrac{3^{x-1} \times (3^{-1})^{x+1}}{(3^{-x})^{-1} \times (3^{x-2})^{-1}}$.

10. $\left\{\dfrac{a^{-\frac{1}{2}}b^{-\frac{1}{3}}}{a^{-\frac{3}{4}}b^{-\frac{5}{6}}} \div (a^{-8}b^{-5})^{\frac{1}{4}}\right\}^{\frac{2}{7}}$.

11. $\left\{\dfrac{x^{\frac{5}{2}}y^{\frac{4}{3}}}{z^{-\frac{5}{4}}} \times \dfrac{z^4}{x^{-8}y^{-\frac{5}{3}}} \div \dfrac{y^{-2}z^{\frac{1}{4}}}{x^{-\frac{1}{2}}}\right\}^{\frac{1}{6}}$.

12. $[(a^{\frac{1}{2}})^{\frac{2}{3}-\frac{1}{6}} - \frac{3}{4}(a^{\frac{5}{2}}b\,\{a^{-8}b^{-2}\}^{\frac{1}{2}})^{\frac{1}{4}}]^4$.

13. $\dfrac{a^{-1}+b^{-1}}{a^{-1}-b^{-1}} \times \dfrac{a^{-2}-b^{-2}}{a^{-2}+b^{-2}} \times \dfrac{1}{\left(1+\frac{b}{a}\right)^{-2} + \left(1+\frac{a}{b}\right)^{-2}}$.

14. $\left\{\dfrac{(x^{-1})^{\frac{1}{4}}}{y^{\frac{4}{3}}} \times \left(\dfrac{x^{\frac{1}{3}}}{y^{-\frac{1}{2}}}\right)^2 \div \dfrac{y^{-\frac{2}{3}}}{x^{-\frac{1}{2}}}\right\}^{12}$.

15. $\left\{\left(\frac{a^{-\frac{1}{2}}b^2}{b^{\frac{1}{2}}a}\right)^{\frac{1}{3}} \times \left(\frac{a^{-1}b}{(ab^2)^{\frac{1}{2}}}\right)^{\frac{1}{3}}\right\}^{-2}$.

16. $\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{c^{\frac{1}{6}}} \div \left\{\frac{c^{-\frac{1}{2}}}{a^{-\frac{1}{3}}b^{-\frac{1}{3}}} \times \frac{a^{-\frac{5}{6}}c^{-\frac{2}{3}}}{b^{\frac{5}{6}}}\right\}$.

XV. RADICALS.

DEFINITIONS.

132. A **Radical Quantity,** or simply a **Radical,** is a root of an expression or number indicated by a radical sign. Thus, $\sqrt{2}$, $\sqrt{a+b}$, and $\sqrt[3]{a-b}$ are radicals in the common form.

The same thing is indicated by fractional exponents; as, $2^{\frac{1}{2}}$, $(a+b)^{\frac{1}{2}}$, and $(a-b)^{\frac{1}{3}}$.

If the indicated root can be exactly found, it is called a *rational quantity;* if it cannot be exactly found, it is called a *surd.*

133. The **Degree** of a radical is indicated by the index of the radical sign (§ 20), and by the denominator of the fractional exponent (§ 13).

134. **Similar Radicals** are such as have the same degree and the same quantity under their radical signs.

REDUCTION OF RADICALS.

TO SIMPLEST FORM.

135. A radical is in its *simplest form* when the quantity under the radical sign is in the integral form, contains no factor whose power is of the same degree as the index of the radical, and is not itself a perfect power whose exponent is a factor of that index.

CASE I.

136. When the quantity under the radical sign contains a factor which is a perfect power of the same degree as the radical.

Reduce the following to their simplest forms:

1. $5\sqrt{20}$.
2. $\frac{1}{2}\sqrt[3]{32}$.
3. $2\sqrt{80}$.
4. $2a\sqrt{28b^2}$.
5. $\sqrt[3]{432}$.
6. $\sqrt{288}$.
7. $\sqrt{1350}$.
8. $\sqrt{6125}$.
9. $\sqrt[3]{-1029}$.
10. $\sqrt{2880}$.
11. $\sqrt{127008}$.
12. $\sqrt{a^3+4a^2b+4ab^2}$.
13. $\sqrt[3]{216-648x^2}$.
14. $\sqrt{8x^3-24x^2y+18xy^2}$.
15. $\sqrt{3x^2y+12xy+12y}$.
16. $\sqrt{x^3-x^2y-xy^2+y^3}$.
17. $\sqrt{x^4-x^3y-xy^3+y^4}$.
18. $3\sqrt[3]{3x^4y-9x^3y^2+9x^2y^3-3xy^4}$.
19. $\sqrt{x^4-x^3y-x^2y^2+xy^3}$.
20. $\sqrt[3]{2a^4b-6a^3b^2+6a^2b^3-2ab^4}$.
21. $\sqrt{x^5-2x^3y^2+xy^4+x^4y-2x^2y^3+y^5}$.
22. $\sqrt[3]{27x^4y+81x^2y^3-81x^3y^2-27xy^4}$.
23. $2\sqrt[5]{32a^5x-64a^5y}$.
24. $\frac{2}{3}\sqrt{3ax^2-30ax+75a}$.
25. $2\sqrt{\frac{7}{9}}$.
26. $3\sqrt{\frac{1}{2}}$.
27. $5\sqrt{\frac{2}{3}}$.
28. $\sqrt{\frac{3}{4}}$.
29. $\sqrt[3]{\frac{2}{3}}$.
30. $5\sqrt[3]{\frac{1}{2}}$.
31. $\sqrt{\dfrac{a-b}{a+b}}$.
32. $\dfrac{5ab}{2}\sqrt{\dfrac{20}{25a^2b}}$.
33. $\dfrac{4a}{x}\sqrt[3]{\dfrac{27x^4}{4a^2}}$.

34. $(x-y)\sqrt{\frac{x+y}{(x-y)^3}}$.

35. $\frac{ab}{x^2-y^2}\sqrt{\frac{(x-y)^3}{ab}}$.

36. $\frac{x^2-y^2}{3x-2y}\sqrt{\frac{27x^2-36xy+12y^2}{x-y}}$.

37. $2\sqrt{\frac{9(a-b)}{2(a+b)}}$.

38. $\sqrt[n]{x^{3n+2}y^{2n+5}}$.

39. $\frac{2y^2}{x^n}\sqrt{\frac{x^{2n+1}}{y}}$.

40. $\frac{b^2}{a^2}\sqrt[n]{\frac{a^{n+1}}{b^{n+1}}}$.

Case II.

137. When the quantity under the radical sign is a perfect power whose exponent is a factor of the index of the radical.

Reduce the following to their simplest forms:

1. $\sqrt[4]{(a+b)^2}$.
2. $\sqrt[6]{8}$.
3. $\sqrt[10]{(a+b)^5x^{10}y^5}$.
4. $\sqrt[12]{64a^6b^{18}}$.
5. $\sqrt[2m]{4a^{2m}b^{4m}c^2}$.
6. $\sqrt[3m+3]{8x^3y^{3m}}$.
7. $\sqrt[4]{16a^{4m}b^{-8m}}$.
8. $\sqrt[6]{216a^{-3}}$.
9. $\sqrt[15]{343a^3b^{-6}}$.
10. $\sqrt[4]{x-2x^{\frac{1}{2}}y^{\frac{1}{2}}+y}$.
11. $\sqrt[n]{a^{2n}b^{2n}c^{mn}}$.
12. $\sqrt[6]{64a^8b^{10}}$.
13. $\sqrt[6]{36x^2+120xy+100y^2}$.
14. $\sqrt[9]{(x-y)^6(x+y)^3}$.
15. $\sqrt[6]{8a^3x^3-48a^2x^3+96ax^3-64x^3}$.
16. $\sqrt[12]{4a^2-24ax+36x^2}$.

TO DIFFERENT FORMS.

CASE I.

138. Introduction of coefficients under the radical sign.

(The converse of § 136.)

Introduce the coefficients under the radical signs in the following:

1. $5a^2\sqrt{a}$.
2. $-2a\sqrt{3x}$.
3. $(x-y)\sqrt{\frac{1}{x-y}}$.
4. $\frac{1}{2}\sqrt{8}$.
5. $\frac{x-y}{x+y}\sqrt{\frac{x+y}{x-y}}$.
6. $\frac{1}{x-y}\sqrt[3]{x^3-y^3}$.
7. $2ax\sqrt[4]{\frac{1}{4a^2x^2}}$.
8. $\frac{1}{x+5}\sqrt{x^2+8x+15}$.
9. $(x+y)\sqrt{\frac{x-y}{x^3+y^3}}$.

CASE II.

139. Reduction to the form of a radical of any degree.

(The converse of § 137.)

Reduce the following to radicals of the required degree:

1. $2ab$ to the 4th.
2. $3a^2b^2$ to the 3d.
3. $\sqrt{a+b}$ to the 10th.
4. $\sqrt[3]{12x^2}$ to the 6th.
5. $\sqrt[4]{4(a-x)}$ to the 8th.
6. $2(a+x)$ to the 3d.
7. $(a+b)^2$ to the mth.
8. $\sqrt[3]{(a+b)^2(a-b)^2}$ to the 6th.
9. $\sqrt[n]{(a+b)^2+c}$ to the mth.

CASE III.

140. Reduction of radicals of different degrees to equivalent radicals of the same degree.

Reduce to equivalent radicals of the same degree:

1. $\sqrt[3]{3}, \sqrt{2}$.
2. $\sqrt{3}, \sqrt[4]{5}$.
3. $\sqrt{2}, \sqrt[3]{3}, \sqrt[4]{3}$.
4. $\sqrt[m]{a}, \sqrt[n]{b}$.
5. $\sqrt[3]{a^2}, \sqrt[4]{b^3}, \sqrt[5]{c^4}$.
6. $\sqrt[5]{\frac{2}{3}}, \sqrt[15]{\frac{1}{2}}$.
7. $\sqrt{a}, \sqrt[3]{b}, \sqrt{b}, \sqrt[3]{c^2}, \sqrt[m]{c^{\frac{3m}{2}}}$.
8. $(a+b)^{\frac{1}{m}}, (b+c)^{\frac{m}{n}}, (c+d)^{\frac{m}{2n}}$.

Arrange in order of magnitude:

9. $\sqrt{3}, \sqrt[3]{4}, \sqrt[4]{7}$.
10. $\sqrt[4]{5}, \sqrt[3]{6}, \sqrt{3}$.
11. $3\sqrt[3]{5}, \sqrt[6]{500}, 2\sqrt{8}$.
12. $12\sqrt[4]{3}, 5\sqrt{6}, 4\sqrt{5}$.
13. $2\sqrt{3}, 3\sqrt[3]{2}, 2\sqrt[3]{6}$.
14. $4\sqrt[3]{2}, 3\sqrt[3]{7}, 4\sqrt{2}$.

141. ADDITION AND SUBTRACTION OF RADICALS.

Combine the following:

1. $4\sqrt{72} + 3\sqrt{32} + 4\sqrt{8} + 2\sqrt{18}$.
2. $3\sqrt{80} + \sqrt{45} + \sqrt{245} + 2\sqrt{45}$.
3. $2\sqrt[3]{54} + \sqrt[3]{250} + 4\sqrt[3]{16} + 3\sqrt[3]{54}$.
4. $2\sqrt{160} + 3\sqrt{90} + \sqrt{490} + 25\sqrt{10}$.
5. $2\sqrt[5]{486} + 4\sqrt[5]{64} + 2\sqrt[5]{15552} + 4\sqrt[5]{2048}$.
6. $\frac{1}{2}\sqrt{75} - \frac{1}{4}\sqrt{27} - \frac{1}{6}\sqrt{3} + \frac{1}{3}\sqrt{12}$.
7. $\frac{1}{2}\sqrt{242} + \frac{4}{3}\sqrt{50} - \frac{1}{5}\sqrt{3362} + \sqrt{\frac{1}{2}}$.
8. $3\sqrt{52} - \sqrt{637} + \frac{1}{3}\sqrt{52} + \frac{1}{3}\sqrt{2197}$.
9. $\frac{1}{2}\sqrt{\frac{3}{4}} + \frac{1}{2}\sqrt{\frac{1}{3}} - \frac{1}{2}\sqrt{\frac{3}{25}}$.

10. $2\frac{2}{3}\sqrt{\frac{5}{4}} + 5\sqrt{\frac{5}{16}} + 3\sqrt{\frac{45}{16}}$.

11. $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{8}} + \frac{1}{4}\sqrt{18}$.

12. $\sqrt[4]{\frac{1}{3}} + \frac{1}{3}\sqrt[4]{\frac{1}{48}} + \sqrt[4]{\frac{32}{486}}$.

13. $5\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{8}} - \sqrt{\frac{4}{18}}$.

14. $2\sqrt{m^2n^2x} + 3m\sqrt{x^3} + m\sqrt{25x^5}$.

15. $2\sqrt{ax^2 - 4a^2x + 4a^3} + \sqrt{16a^3}$.

16. $(8a^3 - 24a^2 + 18a)^{\frac{1}{2}} - (2a^3 - 12a^2 + 18a)^{\frac{1}{2}}$.

17. $\sqrt{(a+b)(a-b)(a+b)^{-2}} + \sqrt{\dfrac{a+b}{a-b}}$.

18. $\dfrac{x-y}{x+y}\sqrt{\dfrac{x+y}{x-y}} + \sqrt{9(x^2-y^2)(x+y)^{-2}}$.

19. $\sqrt[n]{\dfrac{xy}{(x+y)^{-n}}} + \sqrt[n]{\dfrac{(x-y)^n}{(xy)^{-1}}}$.

142. MULTIPLICATION OF RADICALS.

Simplify the following:

1. $5\sqrt{12} \times 3\sqrt{8}$.

2. $2\sqrt[3]{32} \times 5\sqrt[3]{12}$.

3. $\frac{1}{2}\sqrt{20} \times \frac{1}{3}\sqrt{18}$.

4. $\frac{2}{3}\sqrt[3]{54} \times 3\sqrt[3]{64}$.

5. $a^2b\sqrt{ab^2} \times ab^2\sqrt{a^2b}$.

6. $\frac{1}{2}\sqrt[4]{8} \times \frac{1}{3}\sqrt[4]{160}$.

7. $(2\sqrt{8} + 2\sqrt{18})(\sqrt{75} + 4\sqrt{24})$.

8. $(9\sqrt{162} + 3\sqrt{50} + 5\sqrt{2} - 3\sqrt{32})\sqrt{2}$.

9. $5\sqrt{12} \times 3\sqrt{8} \times 6\sqrt{2} \times \sqrt{3}$.

10. $\sqrt[5]{16} \times \sqrt[5]{2} \times \sqrt[5]{96} \times 2\sqrt[5]{12}$.

11. $(3\sqrt{45} - \sqrt{20} + 7\sqrt{5})(4\sqrt{63} + 5\sqrt{7} - 8\sqrt{28})$.

12. $(2\sqrt{99} - 5\sqrt{176} + \sqrt{44})(2\sqrt{363} - 5\sqrt{243}$
$+ \sqrt{192})$.

13. $(3\sqrt[3]{875} - 7\sqrt[3]{56} + 2\sqrt[3]{189})(5\sqrt[3]{81} - 7\sqrt[3]{192}$
$+ 4\sqrt[3]{648})$.

14. $(4\sqrt[4]{32} + 32\sqrt[4]{2} - 2\sqrt[4]{162})(3\sqrt[4]{162} + 7\sqrt[4]{32} + \sqrt[4]{1250})$.

15. $(2\sqrt{8} + \sqrt{\frac{9}{2}} + 6\sqrt{18})(\sqrt{\frac{25}{3}} + \sqrt{12} - \sqrt{\frac{4}{3}})$.

16. $(\sqrt{25a} + 3\sqrt{ab^2} + 2\sqrt{9a})(11\sqrt{a} + b\sqrt{9a} - 3\sqrt{4a})$.

17. $\sqrt{50}\,(8 - \sqrt{39})^{\frac{1}{2}}(8 + \sqrt{39})^{\frac{1}{2}}$.

18. $(3\sqrt{80} - 4\sqrt{20} - \sqrt{45})^2$.

19. $(\sqrt{x+1} + 2)^2$.

20. $(\sqrt{x+1} + \sqrt{x-1})^2$.

21. $(5\sqrt{x} - 2\sqrt{x-1})^2$.

22. $(\frac{1}{2}\sqrt{x} - 12)^2$.

23. $\sqrt{5 - \sqrt{21}} \times \sqrt{5 + \sqrt{21}}$.

24. $\sqrt{6 - 3\sqrt{2}} \times \sqrt{6 + 3\sqrt{2}}$.

25. $(2\sqrt{5} - 3)(4\sqrt{5} + 7)$.

26. $(3\sqrt{7} + 2\sqrt{3})(3\sqrt{7} - 5\sqrt{3})$.

27. $(2\sqrt{2} - \sqrt{3} + \sqrt{5})(3\sqrt{2} + 2\sqrt{3} + 2\sqrt{5})$.

28. $(6\sqrt{6} - 2\sqrt{2} - 8\sqrt{3})(3\sqrt{6} - \sqrt{2} + 4\sqrt{3})$.

29. $\sqrt{15} \times \sqrt[3]{18}$.

30. $\sqrt{3} \times \sqrt[3]{4} \times \sqrt[4]{12}$.

31. $\sqrt{a^2b} \times \sqrt[3]{ab} \times \sqrt[5]{ab^2}$.

32. $\sqrt[3]{a+b} \times \sqrt{a-b} \times \sqrt[4]{a+b}$.

33. $2\sqrt{5} \times 5\sqrt[3]{10} \times 3\sqrt[6]{3}$.

34. $\sqrt[3]{25} \times \sqrt[3]{50} \times \sqrt[4]{15}$.

35. $\sqrt[4]{25} \times \sqrt[6]{27} \times \sqrt[8]{16}$.

36. $\sqrt[5]{64} \times \sqrt[10]{9} \times 3\sqrt[15]{27}$.

37. $\sqrt[5]{a^2x} \times \sqrt[6]{ax} \times \sqrt[6]{a^2x^2}$.

38. $\sqrt[3]{a^3 - x^3} \times \sqrt{a^2 - x^2} \times \sqrt{a - x}$.

39. $\sqrt{a} \times \sqrt[3]{a} \times \sqrt[4]{a} \times \sqrt[5]{a}$.

40. $\sqrt[3]{\frac{3}{4}} \times \sqrt{\frac{1}{2}}$.

41. $\sqrt[5]{\frac{27}{128}} \times \sqrt[8]{\frac{16}{9}}$.

42. $\sqrt[9]{\frac{27}{16}} \times \sqrt[4]{\frac{2}{3}}$.

43. $\sqrt[5]{\frac{25}{16}} \times \sqrt{\frac{2}{5}}$.

44. $\sqrt[4]{\frac{16}{81}} \times \sqrt[3]{\frac{9}{4}}$.

143. DIVISION OF RADICALS.

Simplify the following:

1. $8\sqrt{243} \div 2\sqrt{81}$.
2. $\frac{1}{2}\sqrt{225} \div \sqrt{\frac{5}{4}}$.
3. $\frac{1}{3}\sqrt{\frac{6400}{9}} \div \sqrt{\frac{20}{9}}$.
4. $(3\sqrt{8} + 5\sqrt{2}) \div 3\sqrt{2}$.
5. $(72\sqrt{20} + 32\sqrt{45}) \div 2\sqrt{5}$.
6. $\sqrt{\frac{3}{11}} \div \sqrt{\frac{3}{2}}$.
7. $\sqrt{\frac{2}{3}} \div \sqrt{\frac{25}{3}}$.
8. $3\sqrt{a^2bc} \div \sqrt{4\,a^5bcx}$.
9. $\sqrt{\frac{1}{2}} \div \sqrt[3]{\frac{2}{3}}$.
10. $\sqrt{12} \div \sqrt[4]{10}$.
11. $5\sqrt[3]{128} \div 2\sqrt{432}$.
12. $\sqrt{\frac{1}{6}} \div \sqrt[3]{\frac{1}{18}}$.
13. $(2\sqrt{8} \times 3\sqrt{3}) \div (\sqrt{45} \times 2\sqrt{32})$.
14. $\sqrt[3]{8\frac{1}{6}} \div \sqrt{3\frac{1}{2}}$.
15. $(\sqrt{12} + \sqrt{300}) \div (\sqrt{8} + \sqrt{32})$.
16. $\sqrt{a^3 - b^3} \div \sqrt{(a - b)^3}$.
17. $16 \div \sqrt[3]{768}$.
18. $\sqrt[5]{1600} \div 2$.
19. $\sqrt[3]{64} \div \sqrt{8}$.
20. $\sqrt[3]{128} \div \sqrt{\frac{1}{2}}$.
21. $\dfrac{3\sqrt{48}}{5\sqrt{112}} \div \dfrac{6\sqrt{84}}{\sqrt{392}}$.
22. $\dfrac{3\sqrt{5}}{2\sqrt{98}} \div \dfrac{15}{7\sqrt{15}}$.
23. $\sqrt[4]{\frac{9}{4}} \div \sqrt[9]{\frac{27}{64}}$.
24. $\sqrt[3]{\frac{2}{3}} \div \sqrt[10]{\frac{32}{243}}$.
25. $\sqrt[5]{\frac{16}{27}} \div \sqrt[3]{\frac{4}{9}}$.

144. INVOLUTION OF RADICALS.

Expand the following:

1. $(\sqrt[4]{12})^3$.
2. $(\sqrt{6})^4$.
3. $(\sqrt[3]{15})^4$.
4. $(\sqrt[12]{64})^3$.
5. $(5\sqrt{3a^2bc})^3$.
6. $(\sqrt[3]{a^2-b^2})^{12}$.
7. $(\sqrt{(a+b)^3})^4$.
8. $(\sqrt[3]{12})^4$.
9. $(216^{\frac{1}{2}})^3$.
10. $(-3\sqrt{32})^2$.
11. $(a\sqrt[3]{a^2b^2x})^4$.
12. $(2\sqrt[5]{128})^2$.
13. $(-\sqrt{3a^{\frac{1}{2}}})^3$.
14. $(5\sqrt{2(a^2+4ax+4x^2)})^3$.
15. $(ab\sqrt{3ab})^5$.

145. EVOLUTION OF RADICALS.

Simplify the following:

1. $\sqrt[3]{(5\sqrt{5})}$.
2. $\sqrt[5]{(32\sqrt{32})}$.
3. $\sqrt{(6\sqrt[3]{6^4})}$.
4. $\sqrt[5]{(4\sqrt[3]{4^2})}$.
5. $\sqrt[3]{(\sqrt{27a^3})}$.
6. $\sqrt[5]{(\sqrt[3]{243a^5})}$.
7. $\sqrt[3]{(\sqrt{(a+b)^{12}})}$.
8. $\sqrt{(2\sqrt{216})}$.
9. $\sqrt[6]{(8\sqrt{a^{12}})}$.
10. $5\sqrt[5]{(\sqrt{8\times 128})}$.
11. $3\sqrt[3]{(a^2\sqrt{a^5})}$.
12. $\sqrt{(\sqrt{\sqrt{512x^3}})}$.
13. $\sqrt[3]{(\sqrt{(a+b)^3})}$.
14. $\sqrt{(a+b)^{\frac{3}{2}}}$.
15. $\sqrt[5]{(\sqrt{(a-b)^6})}$.
16. $\sqrt{(\sqrt[3]{m^2n^2x^4})}$.
17. $\sqrt{(3\sqrt[3]{81})}$.
18. $[(a^5b^5)^{\frac{1}{2}}]^{\frac{1}{15}}$.

RATIONALIZATION.

146. Definition. — **Rationalization** is the process of clearing an expression of radicals.

EXERCISES.

Reduce the following to equivalent fractions having rational denominators:

1. $\frac{1}{\sqrt{2}}$.

2. $\frac{2}{\sqrt[4]{2}}$.

3. $\frac{8}{\sqrt[4]{64}}$.

4. $\frac{2}{\sqrt[4]{8}}$.

5. $\frac{1}{\sqrt[3]{2}}$.

6. $\frac{x}{\sqrt{y}}$.

7. $\frac{\sqrt{x-y}}{\sqrt{x+y}}$.

8. $\sqrt{\frac{a}{b}}$.

9. $\frac{1}{\sqrt{a^3}}$.

10. $\frac{\sqrt{5}}{\sqrt{10}}$.

11. $\frac{2}{3\sqrt[3]{a^2}}$.

12. $\frac{3}{\sqrt{12}}$.

13. $\frac{\sqrt{3}-1}{\sqrt{3}+1}$.

14. $\frac{2\sqrt{5}}{\sqrt{5}+\sqrt{3}}$.

15. $\frac{\sqrt{7}+\sqrt{2}}{2\sqrt{14}+9}$.

16. $\frac{3\sqrt{2}-1}{3\sqrt{2}+1}$.

17. $\frac{29}{11+3\sqrt{7}}$.

18. $\frac{17}{3\sqrt{7}+2\sqrt{3}}$.

19. $\frac{10\sqrt{6}-2\sqrt{7}}{3\sqrt{6}+2\sqrt{7}}$.

20. $\frac{2\sqrt{3}+3\sqrt{2}}{5+2\sqrt{6}}$.

21. $\frac{23\sqrt{3}-4\sqrt{2}}{7\sqrt{3}-5\sqrt{2}}$.

22. $\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$.

23. $\frac{\sqrt{9+x^2}-3}{\sqrt{9+x^2}+3}$.

24. $\frac{2\sqrt{x+5}+3\sqrt{x-5}}{2\sqrt{x+5}-\sqrt{x-5}}$.

25. $\frac{(3+\sqrt{5})(\sqrt{5}-2)}{5-\sqrt{5}}$.

26. $\frac{\sqrt{x+2}+\sqrt{x-2}}{\sqrt{x+2}-\sqrt{x-2}}$.

27. $\frac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

28. $\frac{1}{\sqrt{3}+\sqrt{5}-\sqrt{2}}$.

29. $\frac{1}{\sqrt{2}+\sqrt{3}-5}$.

30. $\frac{4+\sqrt{2}-\sqrt{3}}{3-\sqrt{2}+\sqrt{3}}$.

31. $\frac{7\sqrt{15}-3\sqrt{2}}{6\sqrt{3}-2\sqrt{12}+\sqrt{2}}$.

32. $\dfrac{1}{\sqrt{6}+\sqrt{21}-\sqrt{10}-\sqrt{35}}$.

33. $\dfrac{1}{\sqrt{10}+\sqrt{14}+\sqrt{15}+\sqrt{21}}$.

34. $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$.

Simplify the following:

35. $\dfrac{2\sqrt{15}+8}{5+\sqrt{15}}+\dfrac{8\sqrt{3}-6\sqrt{5}}{5\sqrt{3}-3\sqrt{5}}$.

36. $\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}-\dfrac{x-\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}$.

37. $\dfrac{(x+\sqrt{x^2-a^2})^2-(x-\sqrt{x^2-a^2})^2}{(x-\sqrt{x^2-a^2})(x+\sqrt{x^2-a^2})}$

38. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}-\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$.

39. $\left(\sqrt{\dfrac{a+x}{a-x}}-\sqrt{\dfrac{a-x}{a+x}}\right)\sqrt{\dfrac{a^3-x^3}{(a+x)^2-ax}}$.

40. $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$.

Find the values to three decimal places of the following:

41. $\dfrac{1}{\sqrt{2}-1}$.

42. $\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$.

43. $\dfrac{2-\sqrt{3}}{3-\sqrt{2}}$.

44. $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$.

45. $\dfrac{1}{\sqrt{8}+\sqrt{3}}+\dfrac{1}{\sqrt{8}-\sqrt{3}}$.

46. $\dfrac{7+3\sqrt{5}}{7-3\sqrt{5}}+\dfrac{7-3\sqrt{5}}{7+3\sqrt{5}}$.

What factors will rationalize each of the following?

47. $x - y^{\frac{1}{2}}$. 49. $x^{\frac{2}{3}} + y^{\frac{2}{3}}$. 51. $x^{\frac{3}{5}} - y^{\frac{2}{3}}$.

48. $1 + 2^{\frac{1}{3}}$. 50. $2^{\frac{1}{2}} - 4^{\frac{1}{3}}$. 52. $x^{\frac{3}{4}} - y^{\frac{2}{3}}$.

IMAGINARY QUANTITIES.

147. Definition. — An **Imaginary Quantity** is an indicated even root of a negative quantity.

EXERCISES.

Simplify the following:

1. $(3 + 2\sqrt{-1}) + (4 - 2\sqrt{-1}) + (7 + 3\sqrt{-1})$.
2. $\sqrt{-25} + \sqrt{-49} + \sqrt{-121} - \sqrt{-64}$.
3. $(5 + \sqrt{-16}) + (3 - \sqrt{-4}) + (8 + \sqrt{-4})$.
4. $(2\sqrt{-3} + 3\sqrt{-2})(4\sqrt{-3} - 5\sqrt{-2})$.
5. $(3\sqrt{-7} - 5\sqrt{-2})(3\sqrt{-7} + 5\sqrt{-2})$.
6. $(5 - \sqrt{-3} - \sqrt{-7})(5 + \sqrt{-3} + \sqrt{-7})$.

7. $(\sqrt{-3} - \sqrt{-2})^2$. 9. $(5\sqrt{-5} + \sqrt{-1})^2$.

8. $(2\sqrt{-3} - 3\sqrt{-2})^2$. 10. $(2\sqrt{-3} - \sqrt{-1})^2$.

11. $(\sqrt{-2} + \sqrt{-3})^2$.
12. $(\sqrt{-2} + \sqrt{-3} + \sqrt{-1})(\sqrt{-2} + \sqrt{-3} - \sqrt{-1})$.
13. $(\sqrt{-3} - \sqrt{2} + \sqrt{-2})(\sqrt{3} + \sqrt{-2} - \sqrt{2})$.
14. $(2\sqrt{-2} + \sqrt{-3}) \div (2\sqrt{-2} - \sqrt{-3})$.
15. $\dfrac{7 + \sqrt{-3}}{2 - \sqrt{-3}} + \dfrac{8 + 3\sqrt{-3}}{2 + \sqrt{-3}} - \dfrac{4(2 - \sqrt{-3})}{1 - \sqrt{-3}}$.
16. $\dfrac{3 + 2\sqrt{-1}}{2 - 5\sqrt{-1}} + \dfrac{3 - 2\sqrt{-1}}{2 + 5\sqrt{-1}}$.
17. $\dfrac{3\sqrt{-2} + 2\sqrt{-5}}{3\sqrt{-2} - 2\sqrt{-5}}$.

18. $\dfrac{a+x\sqrt{-1}}{a-x\sqrt{-1}}-\dfrac{a-x\sqrt{-1}}{a+x\sqrt{-1}}$.

19. $\dfrac{(x+\sqrt{-1})^2}{x-\sqrt{-1}}-\dfrac{(x-\sqrt{-1})^2}{x+\sqrt{-1}}$.

20. $\dfrac{3+2\sqrt{-18}-\sqrt{-2}}{2+\sqrt{-2}}$.

21. $\left(\dfrac{\sqrt{-5}+\sqrt{-3}}{\sqrt{-5}-\sqrt{-3}}\right)^4-\left(\dfrac{\sqrt{-5}-\sqrt{-3}}{\sqrt{-5}+\sqrt{-3}}\right)^4$.

BINOMIAL SURDS.

148. DEFINITION. — A **Binomial Surd** is a binomial in which one or both of the terms are surds.

EXERCISES.

Extract the square roots of the following:

1. $14+6\sqrt{5}$.
2. $16-6\sqrt{7}$.
3. $6+\sqrt{20}$.
4. $5-\sqrt{24}$.
5. $101-28\sqrt{13}$.
6. $280+56\sqrt{21}$.
7. $117-36\sqrt{10}$.
8. $4\frac{1}{2}+2\sqrt{2}$.
9. $15+2\sqrt{56}$.
10. $2\frac{1}{4}+\sqrt{5}$.
11. $8+4\sqrt{3}$.
12. $12\sqrt{-1}-5$.
13. $-11-60\sqrt{-1}$.
14. $-26+6\sqrt{-3}$.
15. $-8\sqrt{-1}$.
16. $(ab)^2-1+2ab\sqrt{-1}$.
17. $\frac{7}{9}+\frac{4}{3}\sqrt{\frac{1}{3}}$.
18. $\frac{7}{2}+\frac{3}{2}\sqrt{5}$.
19. $4-\sqrt{15}$.
20. $6-\sqrt{35}$.
21. $a^2-2b\sqrt{a^2-b^2}$.
22. $-4ab+2(a^2-b^2)\sqrt{-1}$.
23. $2(a^2+b^2)+2\sqrt{a^4+a^2b^2+b^4}$.

24. $(a+b)^2 - 4(a-b)\sqrt{ab}$.

25. $3b^2 + a^2 + \sqrt{6a^2b^2 + a^2}$.

26. $\sqrt{32} - \sqrt{24}$.

27. $\sqrt{27} + 2\sqrt{6}$.

28. $6 - 4\sqrt{3} + \sqrt{16 - 8\sqrt{3}}$.

29. $3\sqrt{5} - \sqrt{2} + \sqrt{7 + 2\sqrt{10}}$.

30. $2x\sqrt{-1}$.

31. $3x - 1 + 2\sqrt{2x^2 + x - 6}$.

32. $2a + 2\sqrt{a^2 - x^2}$.

Extract the fourth roots of the following:

33. $97 - 56\sqrt{3}$.

34. $17 + 12\sqrt{2}$.

35. $56 + 24\sqrt{5}$.

36. $49 - 20\sqrt{6}$.

37. $\frac{3}{2}\sqrt{5} + 3\frac{1}{2}$.

38. $248 + 32\sqrt{60}$.

Simplify the following:

39. $\dfrac{\sqrt{2} + \sqrt{45}}{\sqrt{2} + \sqrt{7 - 2\sqrt{10}}}$.

40. $\dfrac{\sqrt{3 + 2\sqrt{2}} - \sqrt{2}}{\sqrt{3 - 2\sqrt{2}} + \sqrt{2}}$.

41. $\dfrac{\sqrt{5 + 2\sqrt{6}} - \sqrt{5 - 2\sqrt{6}}}{\sqrt{5 + 2\sqrt{6}} + \sqrt{5 - 2\sqrt{6}}}$.

42. $\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{2} + \sqrt{2 + \sqrt{3}}} - \dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{2} - \sqrt{2 + \sqrt{3}}}$.

43. $\dfrac{1}{\sqrt{12 - \sqrt{140}}} - \dfrac{1}{\sqrt{8 - \sqrt{60}}} - \dfrac{2}{\sqrt{10 + \sqrt{84}}}$.

44. $\dfrac{1}{\sqrt{11 - 2\sqrt{30}}} - \dfrac{3}{\sqrt{7 - 2\sqrt{10}}} - \dfrac{4}{\sqrt{8 + 4\sqrt{3}}}$.

149. RADICAL EQUATIONS.

Solve the following equations:

1. $\sqrt{x^2+11}=x+1.$

2. $2\sqrt{x^2-64}=2x-8.$

3. $\sqrt{x+10}=1+\sqrt{x+1}.$

4. $\sqrt{x+5}-1=\sqrt{x}.$

5. $\sqrt{x-4}-\sqrt{x+11}=-3.$

6. $\sqrt{3x-11}+\sqrt{3x}=\sqrt{12x-23}.$

7. $\sqrt{x+14}-\sqrt{3x+10}=0.$

8. $\sqrt{25x-29}-\sqrt{4x-11}=3\sqrt{x}.$

9. $\sqrt{12x-5}+\sqrt{3x-1}-\sqrt{27x-2}=0.$

10. $\sqrt{x+4ab}=2a+\sqrt{x}.$

11. $(\sqrt{x+4})^{\frac{1}{3}}=\sqrt{2}.$

12. $\sqrt[4]{2x+11}=\sqrt{5}.$

13. $\sqrt[4]{x+10}=\sqrt{\sqrt{x+21}-1}.$

14. $\sqrt{x+9}+\sqrt{x-9}=4+\sqrt{34}.$

15. $\sqrt[3]{x^3+15x^2+25\sqrt{9x^2+64}}=x+5.$

16. $\sqrt{43+2\sqrt{3+\sqrt{6x}}}=7.$

17. $\sqrt{\sqrt{\sqrt{9x}+3}+1}=2.$

18. $[(x+a)^2+2ab+b^2]^{\frac{1}{2}}=b-a-x.$

19. $1+2\sqrt{x}=\sqrt{4x+\sqrt{16x+2}}.$

20. $\dfrac{\sqrt{1+x}+\sqrt{x-7}}{\sqrt{1+x}-\sqrt{x-7}}=2.$

21. $\sqrt{x}+\sqrt{9+x}=\dfrac{45}{\sqrt{9+x}}.$

22. $\dfrac{1}{\sqrt{x+1}} + \dfrac{1}{\sqrt{x-1}} = \dfrac{1}{\sqrt{x^2-1}}.$

23. $2\sqrt{x} - \sqrt{4x-3} = \dfrac{1}{\sqrt{4x-3}}.$

24. $\dfrac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}-\sqrt{2-x}} = 2.$

25. $\dfrac{\sqrt{x}+3}{\sqrt{x}+5} = \dfrac{2\sqrt{x}-1}{2\sqrt{x}+1}.$

26. $\dfrac{6\sqrt{x}-7}{\sqrt{x}-1} - 5 = \dfrac{7\sqrt{x}-26}{7\sqrt{x}-21}.$

27. $\dfrac{1}{x+\sqrt{x^2-50}} + \dfrac{1}{x-\sqrt{x^2-50}} = \dfrac{7}{25}.$

28. $\dfrac{1}{x+\sqrt{x^2-1}} + \dfrac{1}{x-\sqrt{x^2-1}} = 6.$

29. $\dfrac{1}{1-x} + \dfrac{1}{1+\sqrt{x}} - \dfrac{1}{1-\sqrt{x}} = 0.$

30. $(x + c\sqrt{4x+2c})^{\frac{1}{2}} = c + \sqrt{x}.$

31. $\sqrt{\dfrac{x}{a}+\dfrac{c}{b}} + \sqrt{\dfrac{x}{a}-\dfrac{c}{b}} = \sqrt{\dfrac{4x}{a}-\dfrac{2c}{b}}.$

32. $(1+x)^{\frac{1}{2}} + x^{\frac{1}{2}} = 2(1+x)^{-\frac{1}{2}}.$

33. $(x+11)^{\frac{1}{2}} + x^{\frac{1}{2}} = 55x^{-\frac{1}{2}}.$

34. $\dfrac{2x-3}{(x-2)^{\frac{1}{2}}+1} = 2(x-2)^{\frac{1}{2}} - 1.$

35. $\sqrt{ax-x} + \sqrt{a-x} = a\sqrt{1-x}.$

36. $\sqrt{1+x+\sqrt{x}} + \sqrt{1+x-\sqrt{x}} = a.$

37. $\sqrt{x+3} + \sqrt{x-6} = 3(1+\sqrt{2}).$

38. $\sqrt{x+2+\sqrt{x^2+2}} + \sqrt{x-2+\sqrt{x^2+2}} = 2.$

39. $\sqrt{x-a} - \sqrt{x-b} = \sqrt{b} - \sqrt{a}.$

40. $\sqrt[3]{x+1}+\sqrt[3]{x+2}+\sqrt[3]{x-3}=0.$

41. $\sqrt[3]{x+a}+\sqrt[3]{x+b}+\sqrt[3]{x+c}=0.$

42. $\sqrt[3]{a+\sqrt{x}}+\sqrt[3]{a-\sqrt{x}}=\sqrt[3]{b}.$

NOTE. — The remaining equations of this section and an occasional one in following sections, are of higher degree than their classification would call for. All such equations can be reduced to the proper degree, and answers (roots) are given for the reduced forms only.

43. $\sqrt{x}+\sqrt{x-\sqrt{1-x}}=1.$

44. $x-7-\sqrt{49+\sqrt{121x^2+x^4}}=0.$

45. $1+\sqrt{1+x}-\sqrt{1+x+\sqrt{1-x}}=0.$

46. $\sqrt{1+x}+\sqrt{1+x+\sqrt{1-x}}=\sqrt{1-x}.$

47. $\sqrt[3]{x+3}=\sqrt[9]{(x+3)(3x^2+2x-21)}.$

48. $\sqrt{a+\sqrt{a^2+x}}\left(\sqrt{a}-\sqrt{x}\right)=\sqrt{x}.$

49. $\dfrac{5x-1}{\sqrt{5x}+1}=1+\dfrac{\sqrt{5x}-1}{2}.$

50. $\dfrac{3x-1}{\sqrt{3x}+1}=7\left(\dfrac{\sqrt{3x}-1}{x-5}\right).$

51. $\dfrac{\sqrt{x}-1}{x-1}=1-\dfrac{\sqrt{x}}{2\sqrt{x}-5}.$

150. TEST EXAMINATIONS IN RADICALS.

A.

1. Simplify

(1) $5\sqrt{40}.$ (3) $\sqrt[5]{4\sqrt{2}}.$

(2) $8\sqrt[3]{\frac{1}{3}}.$ (4) $\sqrt[3]{3\sqrt{3}}.$

(5) $(a+2)\sqrt{2a^3-8a^2+8a}.$

(6) $(2\sqrt{72}-\sqrt{128})(5\sqrt{2}+\sqrt{8}).$

(7) $\dfrac{\sqrt{x-2}-\sqrt{x+2}}{\sqrt{x-2}+\sqrt{x+2}}$.

(8) $\dfrac{13}{\sqrt{37+20\sqrt{3}}}$.

(9) $(\sqrt{5}-\sqrt{2})(\sqrt{2}+1)(\sqrt{2}-1)(\sqrt{5}+\sqrt{2})$.

(10) $3\sqrt{32}+2\sqrt{8}-3\sqrt{\frac{1}{2}}+\frac{1}{2}\sqrt{2}+\sqrt[4]{4}$.

2. Reduce to a common radical index $\sqrt{5}$, $\sqrt[3]{3}$, and $\sqrt[4]{2}$.

3. Solve the equation $\sqrt{x+7}+\sqrt{x+2}=5$.

4. Find the value of $(1+\sqrt{-1})^2+(1-\sqrt{-1})^2$.

B.

1. Simplify

(1) $2\sqrt{32}$.

(2) $5\sqrt{\frac{3}{5}}$.

(3) $\frac{1}{2}\sqrt[3]{\frac{1}{4}}$.

(4) $\sqrt{\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}}$.

(5) $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}-\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$.

(6) $\dfrac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}+\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$.

2. Express with a rational denominator $\dfrac{1}{\sqrt{2}-\sqrt{3}-\sqrt{5}}$.

3. Solve the equation $\sqrt{x+14}+\sqrt{x+7}=7$.

4. If $x=1+2\sqrt{-1}$ and $y=2+\sqrt{-1}$, find the value of $\sqrt[3]{x^2+y^2}$.

C.

1. Simplify

(1) $4\sqrt{147}-\dfrac{10}{\sqrt{3}}-3\sqrt{75}-2\sqrt{\frac{1}{3}}$.

(2) $5+3\sqrt{8}-7\sqrt{6}+3\sqrt{24}-\sqrt{31-10\sqrt{6}}$.

(3) $\sqrt[3]{\frac{3}{4}} \times \sqrt[2]{\frac{8}{9}}$.

(4) $\dfrac{\sqrt{2}+1}{\sqrt{2}-1} - \dfrac{\sqrt{2}-1}{\sqrt{2}+1}$.

(5) $\sqrt{3+\sqrt{5}} + \sqrt{3-\sqrt{5}}$.

(6) $\dfrac{x+y\sqrt{-1}}{x-y\sqrt{-1}} + \dfrac{x-y\sqrt{-1}}{x+y\sqrt{-1}}$.

2. Square $5\sqrt{3}$, $ab\sqrt[3]{a^2b^2}$, and $\sqrt{(a^2-b^2)(a-b)}$.

3. Solve the equation $\sqrt{x+14} + \sqrt{x+2} = 6$.

4. Solve the equation $\sqrt{x} + \sqrt{x+9} = \dfrac{36}{\sqrt{x}}$.

5. If $x = 1 + 2\sqrt{-1}$, find the value of

$$x^3 - x^2 + 3x + 5.$$

XVI. QUADRATIC EQUATIONS.

DEFINITIONS.

151. A **Quadratic Equation** is an equation of the second degree (§ 108).

152. A **Pure Quadratic Equation** is one that contains only the square of the unknown quantity; as, $ax^2 = b$.

153. An **Affected Quadratic Equation** is one that contains both the square and the first power of the unknown quantity; as, $x^2 + bx = c$. The term containing no unknown quantity is called the *absolute term*.

154. PURE QUADRATIC EQUATIONS.

Solve the following equations:

1. $\sqrt{2x+8} + 2\sqrt{x+5} = 2$.

2. $(x+1)^3 - (x-1)^3 = 26$.

3. $x\sqrt{x^2+12} + x\sqrt{x^2+6} = 3$.

4. $\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}=\sqrt{\frac{x}{b}}+\sqrt{\frac{b}{x}}.$

5. $x+\sqrt{a^2+x^2}=\frac{5\,a^2}{\sqrt{a^2+x^2}}.$

6. $\frac{5\,x+3\sqrt{2\,x^2+9}}{5\,x-3\sqrt{2\,x^2+9}}=19.$

7. $\frac{\sqrt{2\,x^2+1}-\sqrt{x^2-3}}{\sqrt{2\,x^2+1}+\sqrt{x^2-3}}=\frac{1}{2}.$

8. $\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}+\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=98.$

9. $\frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}.$

10. $\sqrt{\frac{x^2-16}{x-3}}+\sqrt{x+3}=\frac{7}{\sqrt{x-3}}.$

11. $\frac{1}{m-\sqrt{m^2-x^2}}-\frac{1}{m+\sqrt{m^2-x^2}}=\frac{m}{x^2}.$

12. $\frac{x+a}{x-a}+\frac{x-a}{x+a}=\frac{3\,a+2\,b}{3\,a-2\,b}+\frac{3\,a-2\,b}{3\,a+2\,b}.$

13. $\frac{2x^2+1+x\sqrt{4\,x^2+1}}{2x^2+3+x\sqrt{4\,x^2+1}}=a.$

14. $\frac{x+a+2\,b}{x+a-2\,b}=\frac{b-2\,a+2\,x}{b+2\,a-2\,x}.$

15. $\sqrt{1-x+x^2}-\sqrt{1+x+x^2}=m.$

AFFECTED QUADRATIC EQUATIONS.

155. Affected quadratic equations may be solved by (1) completing the square; (2) transposing all the terms to one member, and factoring; and (3) by substituting into a formula.

Whenever the equation is simple enough to be factored by inspection, the method by factoring is the most rapid;

but the method by completing the square is usually relied upon, and should therefore be thoroughly understood.

156. QUESTIONS.

1. What are the methods of completing the square?

2. Explain the method of solution by factoring.

3. From the equation $ax^2 + bx = c$ obtain the formula often used in solving quadratic equations.

157. NUMERICAL EQUATIONS.

Solve the following equations:

1. $x^2 + 9x = 22.$
2. $x^2 - 6x = 27.$
3. $x^2 - 8x = -12.$
4. $x^2 - 4x = 5.$
5. $x^2 + 5x = -6.$
6. $x^2 - 3x = 130.$
7. $3x^2 + 7x = 6.$
8. $5x^2 + 9x = 2.$
9. $5x - 6x^2 + 1 = 0.$
10. $35x^2 - 3x = 2.$
11. $17x + 2 = -8x^2.$
12. $4x^2 + 8x = -3.$
13. $6x^2 + 8x = 8.$
14. $8x - 15x^2 = 1.$
15. $6x - 2x^2 = 4.$
16. $10x^2 + 12x + 2 = 0.$
17. $4x^2 - 2x = 2.$
18. $9x^2 + 9x = -2.$

158. NUMERICAL FRACTIONAL EQUATIONS.

Reduce to the proper form and solve the following:

1. $\dfrac{x-1}{x} + \dfrac{x}{x-1} = \dfrac{5}{2}.$

2. $\dfrac{x+1}{x+2} + \dfrac{x+2}{x+1} = \dfrac{26}{5}.$

3. $\dfrac{x}{2x-1} + \dfrac{8-x}{x+5} = \dfrac{13}{6}.$

4. $\dfrac{5-x}{x-3} + \dfrac{4}{(x-1)(x-3)} = 0.$

5. $\dfrac{1}{(x-1)(x-3)} - 6 = \dfrac{2}{x-1} - \dfrac{3}{x-3}.$

6. $\dfrac{4x-3}{3x-7}=3+\dfrac{2x-3}{x-1}.$

7. $\dfrac{5x+2}{2x+5}+\dfrac{1}{20}=\dfrac{2x+5}{3(x+2)}.$

8. $\dfrac{2}{3(x-4)}=\dfrac{5}{4(x-2)}+\dfrac{1}{4}.$

9. $\dfrac{x^3+1}{x+1}+\dfrac{x^4-1}{x^2+1}=6.$

10. $\dfrac{8-x}{2}-\dfrac{2x-11}{x-3}=\dfrac{x-2}{6}.$

11. $\dfrac{x}{x^2-2x-15}-\dfrac{7\frac{1}{2}}{x^2+2x-35}=\dfrac{1}{x^2+10x+21}.$

12. $\dfrac{1}{x}+\dfrac{1}{x+4}=\dfrac{1}{x+1}+\dfrac{1}{x+2}.$

13. $\dfrac{x}{x+1}+\dfrac{x+1}{x+2}+\dfrac{x+2}{x+3}=3.$

14. $\dfrac{x}{x+1}+\dfrac{x+1}{x+2}=\dfrac{x-2}{x-1}+\dfrac{x-1}{x}.$

15. $\dfrac{x-1}{x+1}+\dfrac{x+1}{x-1}=\dfrac{5x}{x^2-1}.$

16. $\dfrac{x^2-5x}{x+3}=x-3+\dfrac{1}{x}.$

17. $\dfrac{4}{x+1}+\dfrac{5}{x+2}=\dfrac{12}{x+3}.$

159. LITERAL EQUATIONS.

Reduce to the proper form and solve:

1. $4x^2+4ax=b^2-a^2.$ 3. $x^2-2ax+4ab=2bx.$

2. $x^2+2ab=2ax+b^2.$ 4. $x^2-2ax+8x=16a.$

5. $3x^2-2ax-bx=0.$

6. $(b-c)x^2+(c-a)x+a-b=0.$

7. $(a+b)x^2+cx-a-b-c=0.$

8. $abx^2-(a^2+b^2)x+ab=0.$

9. $(b^2-a^2)(x^2+1)=2x(a^2+b^2).$

10. $x^2+2x(b-c)+c^2=2bc.$

11. $(a^2-b^2)(x^2-1)=4abx.$

12. $x^2-a^2x-b^2x=ab^3-a^3b.$

13. $\frac{x}{a}+\frac{a}{x}=\frac{a}{b}+\frac{b}{a}.$

14. $x+\frac{1}{x}=a+\frac{1}{a}.$

15. $\frac{a}{x-a}+\frac{b}{x-b}=\frac{2c}{x-c}.$

16. $\frac{(a-x)^2+(b-x)^2}{(a-x)(b-x)}=\frac{5}{2}.$

17. $\frac{x-a}{x-b}+\frac{x-b}{a-x}=\frac{m}{n}.$

18. $\frac{mx}{mx+a}+\frac{nx}{nx+a}=\frac{2px}{px+a}.$

19. $\frac{(x+b)^2}{a^2}-\frac{(x+a)^2}{b^2}=\frac{b^4-a^4}{a^2b^2}.$

20. $(a+b)(ax+b)(a-bx)=(a^2x-b^2)(a+bx).$

21. $\frac{(x-a+b)^2}{(x+a-b)^2}=\left(1-\frac{2x}{a}\right)^2+\left(1+\frac{2x}{b}\right)^2-9$

22. $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{c+a}+\frac{1}{c+b}.$

23. $\frac{x}{x+a}+\frac{x}{x+b}=\frac{c}{c+a}+\frac{c}{c+b}.$

24. $\frac{a+c(a+x)}{a+c(a-x)}=\frac{a}{a-2cx}-\frac{a+x}{x}.$

160. EQUATIONS WHICH CONTAIN RADICALS.

Reduce to the proper form and solve:

1. $2\sqrt{x+1}=\sqrt{x+8}+2.$
2. $3\sqrt{x}=16-\sqrt{x-24}.$
3. $\sqrt{x+5}+\sqrt{x+16}=\sqrt{x+101}.$
4. $2\sqrt{x+3}=4+\sqrt{x-2}.$
5. $3\sqrt{x+20}=\sqrt{8x+9}+8.$
6. $\sqrt{x+9}+\sqrt{x-7}=\sqrt{x-12}.$
7. $\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}.$
8. $\sqrt{4x^2+6x-9}-2\sqrt{x^2-x-4}=3.$
9. $2\sqrt{x+95}-5=\sqrt{5x+200}.$
10. $\sqrt{x+6}+\sqrt{x+13}=\sqrt{2x+43}.$
11. $5\sqrt{x+2}+2\sqrt{x+7}=4\sqrt{x+14}.$
12. $\sqrt{\sqrt{\sqrt{9x}+3}-1}=\sqrt[4]{x}.$
13. $\sqrt[3]{\sqrt{25-x}-\sqrt{x-12}}=\sqrt[6]{x-15}.$
14. $\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}=x-5.$
15. $\sqrt{(x-3)(2x-3)}+\sqrt{(x-1)(2x-5)}=\sqrt{2}.$
16. $\sqrt{2x+4}-\sqrt{\frac{x}{2}+6}=1.$
17. $\sqrt{3+x}+\sqrt{3-x}=2\sqrt{x}.$
18. $2\sqrt{x-2}=\sqrt[4]{64(x-2)^3}.$
19. $12\sqrt{\frac{x}{2}}+5\sqrt{\frac{2}{x}}=26\frac{1}{2}.$
20. $\dfrac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}}=\dfrac{x}{a-x}$

21. $\dfrac{x+\sqrt{12a-x}}{x-\sqrt{12a-x}}=\dfrac{\sqrt{a}+1}{\sqrt{a}-1}.$

22. $\dfrac{x^3+1}{x^2-1}=x+\dfrac{\sqrt{6}}{\sqrt{x}}.$

23. $\dfrac{\sqrt{4x}+1}{\sqrt{4x}-1}=\dfrac{\sqrt{x+3}+\sqrt{x}}{\sqrt{x+3}-\sqrt{x}}.$

24. $x+\sqrt{x^2+5x+6}=2\sqrt{x+2}+2\sqrt{x+3}.$

25. $\sqrt{a+x}+\sqrt{b+x}=\sqrt{a+b+2x}.$

26. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}.$

27. $\sqrt{ax+b^2}+\sqrt{bx+a^2}=a-b.$

28. $\sqrt{a-x}+\sqrt{b+x}=\sqrt{2(a+b)}.$

161. FORMATION OF QUADRATIC EQUATIONS WHEN THE ROOTS ARE GIVEN.

Form the equations when the roots are:

1. $3, -2.$

2. $-5, -1.$

3. $\frac{4}{5}, \frac{3}{7}.$

4. $2 \pm \sqrt{3}.$

5. $7 \pm 2\sqrt{5}.$

6. $-5 \pm 2\sqrt{3}.$

162. PROBLEMS PRODUCING QUADRATIC EQUATIONS OF ONE UNKNOWN QUANTITY.

1. Find two consecutive numbers whose product is 462.

2. The difference of the cubes of two consecutive numbers is 217. What are the numbers?

3. A rectangular field contains two acres, and its length is 32 rods greater than its breadth. What are its length and breadth?

4. What is the price of eggs per dozen when one more for six cents reduces the price per dozen 12 cents?

5. A man buys a certain number of pounds of meat for $8. If he had paid two cents per pound more, he would have got 20 pounds less for his money. How many pounds did he buy?

6. A piece of property is sold for $29.76 at a per cent of profit equal to the cost in dollars. What did it cost?

7. A man bought a number of pieces of cloth for $200, and sold them at $8.32 per piece, gaining as much on the whole number as one piece cost him. What was the number of pieces?

8. A man having 7 miles to walk proceeds one mile at a certain rate per hour, and then completes the distance at a rate one mile per hour faster. He finds that he has been half an hour less upon the road than he would have been had the original rate been unchanged. How much time did he use in walking the 7 miles?

9. Two couriers start at the same time for a place 90 miles distant. The first travels one mile per hour faster than the second, and reaches the end of the journey one hour sooner. What was the rate of each?

10. A laborer digs two trenches, one 40 feet longer than the other. He receives for labor upon each as many cents per foot as it is feet in length, and for the whole work $136. What is the length of each trench?

11. The difference between the hypothenuse and the other two sides of a triangle is respectively 8 and 4 feet. Find the other two sides.

12. A certain number consists of two digits whose product is 35, and if 18 is subtracted from the number the order of its digits will be inverted. What is the number?

13. A cistern can be filled by two pipes running together in $3\frac{3}{4}$ hours, but the larger one will fill it alone in four hours less time than the smaller one. What time is required by each to fill it alone?

14. The cost of a dinner was $60, which was to have been divided equally among a certain number of men; but five of them failed to be present, and consequently the others were obliged to pay one dollar each more than they would otherwise have done. What was the original number?

15. A person drew a quantity of wine from a full vessel containing 81 gallons, and filled up the vessel with water. He then drew from the mixture the same amount that he drew the first time, and found that there remained in the vessel 64 gallons of pure wine. How much did he draw each time?

16. A man invests $4000 at a certain rate of interest. At the end of one year he withdraws $100 from the amount due, and leaves the remainder, both principal and interest, upon interest at the same rate. At the end of the second year he finds the amount due him to be $4388.40. What was the rate of interest?

17. A man buys a certain number of $100 shares for $5100, when they are at a certain discount. He keeps ten shares, and sells the remainder for $6500 when the premium is double the original discount. How many shares did he buy?

18. A man walks 6 miles into the country. On his return he walks $22\frac{1}{2}$ minutes, and is then detained the same length of time; he now finds that in order to use no more time in returning than in going out, he must for the remainder of the distance increase his speed two miles per hour. What was his ordinary rate?

XVII. EQUATIONS IN THE QUADRATIC FORM.

DEFINITIONS.

163. A **Cubic Equation** is an equation of the third degree.

164. A **Biquadratic Equation** is an equation of the fourth degree.

165. Equations above the second degree are called **Higher Equations.**

166. An equation is in the *quadratic form* when it is so arranged as to show two powers, one of which is twice the other, of the same quantity, and an absolute term. Thus, Exercises 1–6, § 169; and 1–12, § 170, are in the quadratic form. Such equations can be *first* solved as quadratics, and then a complete solution can be readily obtained.

167. It is evident that the lower power of the quantity is the unknown quantity of the equation considered as a quadratic. This power, with its two values, will give two equations, which must be solved according to their degree and form.

If they take the form of simple equations, or of pure or affected quadratic equations, the methods of solution are known (§§ 111, 154, 155). For forms other than these, some methods will be given in §§ 179, 181. The roots of all equations derived from the original equation will be roots of that equation.

168. Every equation has as many roots as there are units in its degree, but they cannot all be found in every case by the methods of elementary algebra. (See § 108.)

EXERCISES.

169. Reduce to the quadratic form, if necessary, and solve the following:

1. $x^4 - 5x^2 + 4 = 0.$
2. $x^4 - 35x^2 + 216 = 0.$
3. $3x^4 - 7x^2 = 43076.$
4. $3x + 2x^{\frac{1}{2}} - 1 = 0.$
5. $x^{\frac{1}{4}} + 5x^{\frac{1}{2}} = 22.$
6. $3x^{\frac{3}{2}} - 4x^{\frac{3}{4}} = 7.$
7. $6x^{\frac{1}{2}} = 5x^{-\frac{1}{2}} - 13.$
8. $2x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} = 5.$
9. $3x^3 + 42x^{\frac{3}{2}} = 3321.$
10. $x^{\frac{1}{2}} - 3 + 2x^{-\frac{1}{2}} = 0.$
11. $x^{\frac{3}{4}} - 2x^{\frac{1}{2}} + x = 0.$
12. $x^{-2} - 2x^{-1} = 8.$
13. $x^{-4} - 10x^{-2} = -9.$
14. $x^{-1} + 3 = 4x^{-\frac{1}{2}}.$
15. $x^{-\frac{3}{2}} + 8 = 9x^{-\frac{3}{4}}.$
16. $6x^{\frac{3}{4}} = 7x^{\frac{1}{4}} - 2x^{-\frac{1}{4}}.$
17. $x^{2n} + 4x^n = 5.$
18. $x^{\frac{2}{n}} - 5x^{\frac{1}{n}} = -6.$
19. $3x^{\frac{1}{2n}} - x^{\frac{1}{n}} = 2.$

170. An **expression** of one or more terms may take the place of the single unknown quantity.

Solve the following equations:

1. $(x-5)^2+5(x-5)=50.$
2. $(2x^2+1)^2+(2x^2+1)=90.$
3. $(5x-3)^2-9(5x-3)=286.$
4. $(x^2-x)^2-8(x^2-x)=-12.$
5. $(x^2+x)^2-22(x^2+x)=-40.$
6. $\left(\frac{x^2+1}{x}\right)^2+4\left(\frac{x^2+1}{x}\right)=12.$
7. $x+5+2\sqrt{x+5}=15.$
8. $x-2+3\sqrt{x-2}=0.$
9. $5(x+6)+5\sqrt{x+6}=60.$
10. $6x+3=3\sqrt{6x+3}.$
11. $x+5+\sqrt{x+5}=12.$
12. $\sqrt{4x+9}+4x+9=30.$

Reduce the following to the quadratic form and solve:

13. $x^2-4x+2\sqrt{x^2-4x+7}=1.$
14. $x^2-6x+12+\sqrt{x^2-6x+9}=5.$
15. $x^2+5x+4=5\sqrt{x^2+5x+28}.$
16. $2x^2-2x+6\sqrt{x^2-x+7}=22.$
17. $2x^2+6x=226-\sqrt{x^2+3x-8}.$
18. $9\sqrt{x^2-9x+28}+9x=x^2+36.$
19. $x^2+5x-10+2\sqrt{x^2+5x+2}=12.$
20. $x^2-4x+6+4\sqrt{x^2-7x+11}=3x.$
21. $3x^2-7+3\sqrt{3x^2-16x+21}=16x.$

22. $2x^2 - x + 14 - 11\sqrt{2x^2 - 3x + 16} + 30 = 2(x-1)$.

23. $(x-3)^2 - 13 = -\sqrt{x^2 - 6x + 16}$.

24. $2\sqrt{x^2 - 2x + 1} + x^2 = 23 + 2x$.

25. $x^2 - x + 3\sqrt{2x^2 - 3x + 2} = \frac{x}{2} + 7$.

26. $8 + 9\sqrt{(3x-1)(x-2)} = 3x^2 - 7x$.

27. $(x+5)(x-2) + 3\sqrt{x^2 + 3x} = 0$.

28. $\frac{3x-2}{2} + \sqrt{2x^2 - 5x + 3} = \frac{(x+1)^2}{3}$.

29. $\sqrt{x^2 - 2x + 9} - \frac{x^2}{2} = 3 - x$.

30. $x(x+1) + 3\sqrt{2x^2 + 6x + 5} = 25 - 2x$.

31. $x^4 + 2x^3 - 31x^2 - 32x + 60 = 0$.

32. $x^4 + 8x^3 + 24x^2 + 32x + 16 = 0$.

33. $x^4 + 4x^3 + 5x^2 + 2x = 240$.

34. $x^4 - 6x^3 + 10x^2 - 3x = 2$.

35. $x^4 - 8x^3 + 12x^2 + 16x - 12 = 0$.

XVIII. SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

DEFINITIONS.

171. A **Homogeneous Equation** is one in which all the terms are of the same degree (§ 33).

172. A **Symmetrical Equation** of two unknown quantities is one in which the quantities can be interchanged without destroying the equation.

EQUATIONS OF TWO UNKNOWN QUANTITIES.

CASE I.

173. When one equation is of the first degree and the other of the second.

Solve the following equations:

1. $$\begin{cases} x + y = 5. \\ x^2 + 2y^2 = 22. \end{cases}$$

2. $$\begin{cases} x - y = 3. \\ xy = 10. \end{cases}$$

3. $$\begin{cases} x + y = \frac{15}{4}. \\ x - y = xy. \end{cases}$$

4. $$\begin{cases} x + 2y = 7. \\ \dfrac{3}{x} + \dfrac{6}{y} = 5. \end{cases}$$

5. $$\begin{cases} 2x + y = 16. \\ xy + y^2 = 66. \end{cases}$$

6. $$\begin{cases} x + y = 5. \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6}. \end{cases}$$

7. $$\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} = 8. \\ xy = \frac{1}{6}. \end{cases}$$

8. $$\begin{cases} 3x + 2y = 5. \\ 4x^2 + 9y^2 = 13. \end{cases}$$

9. $$\begin{cases} x^2 - y^2 = 5. \\ x + y = 5. \end{cases}$$

10. $$\begin{cases} x + y = -1. \\ x^2 + y^2 = 25. \end{cases}$$

11. $$\begin{cases} x^2 - y^2 = 24. \\ x - 2y = 3. \end{cases}$$

12. $$\begin{cases} x + y = 8. \\ xy = 12. \end{cases}$$

13. $$\begin{cases} x^2 - y^2 = 16. \\ 5x + y = 28. \end{cases}$$

14. $$\begin{cases} x - y = 1. \\ 7x^2 + 5xy - 6y^2 = 32. \end{cases}$$

15. $$\begin{cases} 2x - 3y = -2. \\ x^2 + 3xy + y^2 = 20. \end{cases}$$

16. $$\begin{cases} \dfrac{x}{2} + \dfrac{y}{5} = 5. \\ \dfrac{2}{x} + \dfrac{5}{y} = \dfrac{5}{6}. \end{cases}$$

CASE II.

174. When the equations are both symmetrical, or in forms which can be solved in the same manner as symmetrical equations.

Solve the following equations:

1. $\begin{cases} x^2 + y^2 = 26. \\ xy = 5. \end{cases}$
2. $\begin{cases} x^2 + y^2 = 13. \\ x + y = 5. \end{cases}$
3. $\begin{cases} x - y = 4. \\ 2xy = 42. \end{cases}$
4. $\begin{cases} x + y = 10. \\ \sqrt{xy} = 3. \end{cases}$
5. $\begin{cases} x^2 + xy + y^2 = 49. \\ x + y = 8. \end{cases}$
6. $\begin{cases} x^2 - xy + y^2 = 211. \\ xy = 15. \end{cases}$
7. $\begin{cases} x - y = 2. \\ x^3 - y^3 = 98. \end{cases}$
8. $\begin{cases} x + y = 1. \\ x^3 + y^3 = 61. \end{cases}$
9. $\begin{cases} x - y = 37. \\ x^{\frac{1}{3}} - y^{\frac{1}{3}} = 1. \end{cases}$
10. $\begin{cases} x + y = 35. \\ x^{\frac{1}{3}} + y^{\frac{1}{3}} = 5. \end{cases}$
11. $\begin{cases} x^3 - y^3 = 98. \\ 3xy(x - y) = 90. \end{cases}$
12. $\begin{cases} x - y = 2. \\ x^2y - xy^2 = 30. \end{cases}$
13. $\begin{cases} x^2y + xy^2 = 30. \\ x^3 + y^3 = 35. \end{cases}$
14. $\begin{cases} x^2 + xy + y^2 = 13. \\ x^4 + x^2y^2 + y^4 = 91. \end{cases}$
15. $\begin{cases} x^2 - xy + y^2 = 28. \\ x^4 + x^2y^2 + y^4 = 1456. \end{cases}$
16. $\begin{cases} x^2 + xy + y^2 = 68\frac{1}{4}. \\ x - \sqrt{xy} + y = 6\frac{1}{2}. \end{cases}$
17. $\begin{cases} x^2 + xy + y^2 = 84. \\ x + \sqrt{xy} + y = 14. \end{cases}$
18. $\begin{cases} x^4 + y^4 = 881. \\ xy = 20. \end{cases}$
19. $\begin{cases} x^5 + y^5 = 1056. \\ x + y = 6. \end{cases}$
20. $\begin{cases} x^3 + y^3 = 28xy. \\ x + y = 12. \end{cases}$

CASE III.

175. When both Equations are Homogeneous Quadratics.

Solve the following equations:

1. $\begin{cases} y^2 + xy = 6. \\ x^2 + 3xy = 7. \end{cases}$
2. $\begin{cases} x^2 - 3xy = 10. \\ 4y^2 - xy = -1. \end{cases}$

3. $\begin{cases} x^2+xy=24. \\ 2y^2+3xy=32. \end{cases}$

4. $\begin{cases} x^2-4y^2=9. \\ xy+2y^2=3. \end{cases}$

5. $\begin{cases} x^2+xy=12. \\ xy-2y^2=1. \end{cases}$

6. $\begin{cases} x^2-xy+y^2=21. \\ y^2-2xy=-15. \end{cases}$

7. $\begin{cases} x^2+xy-6y^2=21. \\ xy-2y^2=4. \end{cases}$

8. $\begin{cases} x^2+2y^2=22. \\ 3y^2-xy-x^2=17. \end{cases}$

9. $\begin{cases} x^2+xy+4y^2=6. \\ 3x^2+8y^2=14. \end{cases}$

10. $\begin{cases} 2x^2-3xy+5y^2=64. \\ 3x^2+xy+y^2=36. \end{cases}$

176. MISCELLANEOUS EXERCISES.

Solve the following equations:

1. $\begin{cases} x^2+xy+y^2=127. \\ x^2y^2+xy=1806. \end{cases}$

2. $\begin{cases} (x+y)^2+x+y=56. \\ xy=10. \end{cases}$

3. $\begin{cases} x^2y+xy^2=30. \\ xy+y^2=10. \end{cases}$

4. $\begin{cases} x^4+y^2=25. \\ x^2+y=7. \end{cases}$

5. $\begin{cases} x^4+y^4=706. \\ x+y=8. \end{cases}$

6. $\begin{cases} x^4+y^4=272. \\ x-y=2. \end{cases}$

7. $\begin{cases} x^5-y^5=3093. \\ x-y=3. \end{cases}$

8. $\begin{cases} x^2+y^2=13. \\ x+y+xy=11. \end{cases}$

9. $x+y+3\sqrt{x+y}=x^2+y^2=10.$

10. $\begin{cases} x^2+y^2+2\sqrt{x^2+y^2}=35. \\ x^4+y^4=337. \end{cases}$

11. $\begin{cases} x^2+y^2-x-2y=1. \\ x^2y^2-2x^2y-xy^2+2xy+2=0. \end{cases}$

12. $\begin{cases} x^3-y^3=7xy. \\ x-y=2. \end{cases}$

13. $\begin{cases} \sqrt[3]{x+10}+\sqrt[3]{y+14}=12. \\ x+y=444. \end{cases}$

14. $\begin{cases} \dfrac{\sqrt[3]{x+y}-\sqrt[3]{x-y}}{\sqrt[3]{x+y}+\sqrt[3]{x-y}}=3. \\ x^2+y^2=75. \end{cases}$

15. $\begin{cases} x^4+x^2y^2+y^4=108\,x^2. \\ x^2+xy+y^2=18\,x. \end{cases}$

16. $\begin{cases} \dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}+\dfrac{x}{y}-\dfrac{y}{x}=\frac{106}{9}. \\ xy=3. \end{cases}$

17. $\begin{cases} \left(\dfrac{x}{y}\right)^{\frac{1}{2}}+\left(\dfrac{y}{x}\right)^{\frac{1}{2}}=61\,(xy)^{-\frac{1}{2}}+1. \\ x^{\frac{3}{4}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{3}{4}}=78. \end{cases}$

18. $\begin{cases} xy^{\frac{1}{2}}+x^{\frac{1}{2}}y=20. \\ x^{\frac{3}{2}}+y^{\frac{3}{2}}=65. \end{cases}$

19. $\begin{cases} x^{-3}+y^{-3}=152. \\ x^{-1}+y^{-1}=8. \end{cases}$

20. $\begin{cases} x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}=280. \\ y^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}=945. \end{cases}$

21. $\begin{cases} x^{\frac{2}{3}}+y^{\frac{2}{3}}+4\,(x^{\frac{1}{3}}+y^{\frac{1}{3}})=26. \\ x^{\frac{1}{3}}y^{\frac{1}{3}}=3. \end{cases}$

22. $\begin{cases} x-y=\sqrt{x}+\sqrt{y}. \\ x^{\frac{3}{2}}-y^{\frac{3}{2}}=61. \end{cases}$

EQUATIONS CONTAINING MORE THAN TWO UNKNOWN QUANTITIES.

177. Solve the following equations:

1. $\begin{cases} xy=24. \\ xz=32. \\ yz=12. \end{cases}$

2. $\begin{cases} x\,(x+y+z)=36. \\ y\,(x+y+z)=48. \\ z\,(x+y+z)=60. \end{cases}$

3. $\begin{cases} x\,(y+z)=20. \\ y\,(x+z)=18. \\ z\,(x+y)=8. \end{cases}$

4. $\begin{cases} x+y+z=20. \\ xy=35. \\ yz=56. \end{cases}$

5. $$\left\{\begin{array}{l} x^2 + y^2 + z^2 = 30. \\ x + y + z = 8. \\ yz = 2. \end{array}\right.$$

6. $$\left\{\begin{array}{l} x + y + z = 24. \\ xy + yz = 128. \\ xz = 63. \end{array}\right.$$

178. PROBLEMS PRODUCING SIMULTANEOUS EQUATIONS OF TWO OR MORE UNKNOWN QUANTITIES.

1. The sum of two numbers is 8, and the sum of their cubes is 152. What are the numbers?

2. The sum of two numbers is 280, and the sum of their cube roots is 10. What are the numbers?

3. Four cows and 5 sheep can be bought for $220; but for $480 twenty-eight more sheep than cows can be bought. What is the price of each?

4. Find two numbers, each of which is the square of the other, and whose sum is -1.

5. Find two numbers whose sum, product, and difference of their squares are equal.

6. The joint stock of two partners was $1200. A's money was in trade 15 months and B's 12 months. A received $537.50, capital and profit; and B received $742, capital and profit. What was the capital of each?

7. The fore wheel of a carriage makes 88 more revolutions than the hind wheel in going a mile; but, if the circumference of each wheel is decreased two feet, the fore wheel will make 132 more revolutions than the hind wheel in going the same distance. What is the circumference of each wheel?

8. $200 amounted to $260 at a certain rate and time. If the time had been one year less, and the rate two per cent more, it would have amounted to $264. What were the time and rate?

9. $1500 is invested in two parts and at different rates of interest, so as to give the same income. If the first part were invested at the second rate, the income would be

$64; but if the second part were invested at the first rate, the income would be $49. What are the rates of interest?

10. A man sold 7 cows and 8 sheep for $320, and found that he had sold one cow less for $200 than sheep for $30. What was the price of each?

11. Find three numbers whose sum is 47, the sum of whose squares is 769, and of which the product of the first and third, minus the second, is 225.

12. A certain number consists of three digits whose sum is 9, the sum of whose squares is 29, and the product of the second and third is six times the first. What is the number?

13. The sum of three numbers is 16, the sum of the squares of the first two is 97, and the sum of the three products obtained by multiplying them together in pairs is 75. What are the numbers?

14. Two trains start from two stations at different rates of speed, each train going toward the other station. When they meet one train has travelled 108 miles more than the other, and they finish their trips in 9 and 16 hours respectively. Find the rate per hour of each train, and the distance between the stations.

15. A and B agree to do a piece of work for $30. A can do it alone in 12 days. They agree to do it in 6 days, but after working together for 3 days they hire C, and all three together finish the work in the time agreed upon, in consequence of which A receives $1.66⅔ less than he would if C had not been hired. How long would it have taken B and C, each working alone?

XIX. MISCELLANEOUS QUADRATIC AND HIGHER EQUATIONS.

A METHOD OF FINDING ALL THE ROOTS OF AN EQUATION.

179. In reducing some of the foregoing equations to special forms, certain roots were neglected. (See Note, page 111; and § 168.) Such roots may often be found by the following rule: *Whenever an equation is divided by an expression containing the unknown quantity, such expression must be put equal to 0, and solved as an independent equation. All roots of equations derived in this way will be roots of the original equation.* It will be evident from this rule that the degree of an equation is depressed by the amount of the degree of each divisor. Therefore, whenever by any process all the roots but two are found, by dividing the original equation by the proper factors, equal to 0, formed from these roots, it may be depressed to the quadratic degree, and the two remaining roots found in the ordinary way.

EXERCISES.

Find all the roots in the following:

1. $x + \sqrt{x^2 - \sqrt{1 - 2x}} = 1.$

2. $\sqrt[3]{2x + 1} = \sqrt[9]{(2x + 1)(6x^2 + 7x + 2)}.$

3. $\sqrt{7x^2 - 11x + 6} + \sqrt{6x^2 - 11x + 15} = 2(x + 3).$

4. $\sqrt{7x^2 - 6x - 1} = \sqrt{2x^2 + 5x - 7}$
$+ \sqrt{3(x^2 - 7x + 6)}.$

5. $\dfrac{(x + 1)^3}{x^3 - x^2 - x + 1} = \dfrac{a^2}{b^2}.$

6. $\dfrac{x - a}{b} + \dfrac{x - b}{a} = \dfrac{b}{x - a} + \dfrac{a}{x - b}.$

7. $\sqrt{x(a + b - x)} + \sqrt{a(b + x - a)}$
$+ \sqrt{b(a + x - b)} = 0.$

8. $\sqrt{a(a+b+x)} - \sqrt{a(a+b-x)} = x$.
9. $\sqrt{(a+x)(x+b)} + \sqrt{(a-x)(x-b)} = 2\sqrt{ax}$.
10. $\dfrac{a+2x+\sqrt{a^2-4x^2}}{a+2x-\sqrt{a^2-4x^2}} = \dfrac{5x}{a}$.

Transpose all the terms to the first member, factor, and solve:

11. $x^3 + 2x^2 - 8x = 16$.
12. $x^3 + 6x^2 - 5x = 30$.
13. $x^4 + 4x^3 - 5x^2 - 20x = 0$.
14. $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.
15. $x^4 - 2x^3 + 27x = 54$.
16. $x^5 - x^4 - 64x^2 + 64x = 0$.

FORMATION OF HIGHER EQUATIONS WHEN THE ROOTS ARE GIVEN.

180. The principle of § 161 may be extended to equations of any degree.

EXERCISES.

Form the equations when the roots are:

1. $2, 3, -4$.
2. $2, -5, 3$.
3. $2, -2, -3, 4$.
4. $-2, -2, -2, -2$.
5. $-5, 3 \pm \sqrt{2}$.
6. $2 \pm \sqrt{-1}, \sqrt{2} \pm 3$.

ANOTHER METHOD OF FINDING ALL THE ROOTS.

181. Some equations in the form of Exercises 31–35, § 170, which cannot be reduced to the quadratic form, can be solved by reversing the order of work in § 180, as follows: *Transpose all the terms to the first member, and find by trial what factors of the absolute term will satisfy the equation. These factors will be roots of the equation.* Complete the solution by the last clause of § 179 when necessary.

EXERCISES.

Solve the following equations:

1. $x^3 - 2x + 1 = 0$.
2. $x^3 - 5x + 4 = 0$.
3. $x^3 - 49x + 120 = 0$.

4. $(x-1)(x-2)(x-3)=24.$

5. $(x-2)(x-3)(x-4)=6.$

6. $x^3+x^2-4x-4=0.$

7. $x^4-10x^3+35x^2-50x+24=0.$

8. $x^4+2x^3-11x^2+4x+4=0.$

182. FACTORING.

Some expressions of the quadratic degree can be factored by placing them equal to 0, solving the equations thus formed, and applying the principle involved in § 161 to the roots.

This method is useful when the factoring cannot be readily done by inspection.

EXERCISES.

Factor the following:

1. $x^2-x-72.$

2. $x^2-20x-341.$

3. $x^2-6x-187.$

4.* $5x^2+14x-55.$

5. $4x^2+17x-15.$

6. $18x^2-27x-26.$

7. $x^2-2x-2.$

8. $19+4\sqrt{15}-x^2.$

9. $x^2-2x-1.$

10. $4x^2-12x+7.$

11. $x^2+3xy+2xz+2y^2+3yz+z^2.$

12. $2x^2+3xy+x+y^2-y-6.$

13. $6x^2+xy+9x-2y^2-8y-6.$

14. $21x^2+17xy+10xz+2y^2+3yz+z^2.$

15. $5x^2+6xy+6xz+y^2+2yz+z^2.$

16. $6x^2+7xy-6x-3y^2-20y-12.$

---

* Consider the coefficient of x^2 one factor. The three factors thus obtained can be made into two in the final result. For other easy exercises see §§ 62, 63, 68.

183. TEST EXAMINATIONS IN QUADRATICS.

A.

1. Solve $\frac{5}{x-2}-\frac{4}{x}=\frac{3}{x+6}$.

2. Solve $x^2+ax(1+3b)+3a^2b=0$.

3. Solve $(x^2+x)^2+(x^2+x)=42$.

4. Solve $3x-\sqrt{2x^2+6x+1}=1-x^2$.

5. Solve $\begin{cases} x^{\frac{1}{3}}+y^{\frac{1}{3}}=6. \\ x+y=72. \end{cases}$

6. A number consists of three digits whose sum is 14. The square of the middle digit is equal to the product of the two end digits; and, if 594 is added to the number, the order of the digits will be inverted. What is the number?

B.

1. Solve $\frac{2x}{x-3}-\frac{x-5}{2-x}=\frac{17}{4}$.

2. Solve $ax^2+bx(1-a^2)=ab^2$.

3. Solve $x^3-7x^{\frac{3}{2}}=8$.

4. Solve $x^2+4x+2\sqrt{x^2+4x+11}=13$.

5. Solve $\begin{cases} x+\sqrt{x+y}=12-y. \\ x^2+y^2=41. \end{cases}$

6. Eight hundred persons are seated on benches of equal length. If there were 20 benches less, it would be necessary that two persons more should sit on each bench in order that all might be seated. Required the number of benches.

C.

1. Solve $\frac{4}{x-3}-\frac{3}{x+5}=\frac{17}{10}$.

2. Solve $x^{\frac{2}{3}}+5x^{\frac{1}{3}}=14$.

3. Solve $x + \sqrt{x+4} = 8$.

4. Solve $(a-x)^2 - (a-x)(b-x) + (x-b)^2 = (a-b)^2$.

5. Solve $\begin{cases} x^3 + y^3 = 133. \\ x + y = 7. \end{cases}$

6. Form the equation whose roots shall be $-1 \pm \sqrt{2}$.

XX. INEQUALITIES.

184. Definition.—An **Inequality** is a statement that one algebraic expression is greater or less than another (§ 22).

EXERCISES.

Find the limits of x in the following:

1. $3x + \frac{x}{2} - 12 > \frac{5x}{2} + \frac{2}{3}$.
2. $6x - \frac{2x-3}{4} > 3x - \frac{4x-1}{2}$.
3. $\frac{5x}{6} - \frac{2x+3}{4} - \frac{x^2 - 11x + 30}{x-5} < \frac{3x-2}{5} + \frac{5}{2}$.
4. $\frac{3x}{2} - \frac{\frac{2x-1}{3}}{\frac{3}{4}} + 5 > \frac{x}{3} - \left(4 - \frac{2x-1}{3}\right)$.
5. $\begin{cases} 2x - 5 > 5x - 25. \\ 4x - 1 < 5x - 2. \end{cases}$
6. $\begin{cases} 3x - y > 12. \\ 3y - x > 2. \end{cases}$
7. $\begin{cases} 5x - 6y > 10. \\ 2x + y > 12. \end{cases}$
8. $\begin{cases} 5x - y > 20. \\ 2x - y < 8. \end{cases}$
9. $\begin{cases} 2x + y > 12. \\ x + 2y < 9. \end{cases}$
10. $\begin{cases} 4x - 2y = 6. \\ 3x + y > 16. \end{cases}$

If a, b, and c are positive and unequal, prove:

11. $a^2 + b^2 > 2ab$.

12. $a^4 + b^4 > a^3b + ab^3$.

13. $b^3 + c^3 > bc(b + c)$. **14.** $a^6 + b^6 > a^5b + ab^5$.

15. $(4b - a)(a - 2b) < b^2$.

16. $a^3 + a^2b + ac^2 + bc^2 > 2a^2c + 2abc$.

17. $2(a^3 + b^3 + c^3) > bc(b + c) + ca(c + a) + ab(a + b)$.

What more than in 11–17 must be known to prove the following?

18. $a^3 - b^3 > 3a^2b - 3ab^2$. **19.** $(a - c)^3 < a^3 - c^3$.

XXI. RATIO AND PROPORTION.

DEFINITIONS.

185. **Ratio** is the relative magnitude of two quantities of the same kind, and is found by dividing one by the other (§ 23).

The quantities compared are called the *terms* of the ratio; the first being called the *antecedent*, and the second the *consequent*.

186. A **Proportion** is an equality of two ratios, and must therefore have four terms. The first and third of these are called *antecedents;* and the second and fourth, *consequents.* The first and fourth are called *extremes;* and the second and third, *means.* The first two are called the *first couplet;* and the last two, the *second couplet.* Either of the signs = or : : is used to express the equality.

187. In the proportion $a : b = b : c$, b is called a *mean proportional* between a and c; and c, a *third proportional* to a and b. When all the terms are different the last is called a *fourth proportional* to the other three.

188. EXERCISES IN RATIO.

1. Arrange in order of magnitude the ratios 3 : 4, 5 : 7, 11 : 14.

2. Compare the ratios

$$x + 2y : x - 2y \text{ and } x + y : x + 2y.$$

Find the ratio of x to y in the four following:

3. $8x = 3y$.

4. $ax + by = cx - dy$.

5. $15(2x^2 - y^2) = 7xy$.

6. $6(x^2 + y^2) = 13xy$.

7. If $x : y = 3 : 4$, find the ratio of $7x - 4y : 3x + y$.

8. If $\dfrac{y}{x - z} = \dfrac{y + x}{z} = \dfrac{x}{y}$, find the ratios of $x : y$ and $y : z$.

9. If 14 cows and 12 sheep are worth as much as 5 cows and 48 sheep, what is the ratio of the value of a cow to that of a sheep?

189. EXERCISES IN PROPORTION.

Find a fourth proportional to:

1. 3, 5, and 7.

2. $x^3 + y^3$, $x^2 - xy + y^2$, and $x + y$.

Find a mean proportional between:

3. a^3b and ab^3.

4. $(a + b)^2$ and $(a - b)^2$.

Find a third proportional to:

5. $(a - b)^2$ and $a^2 - b^2$.

6. $\dfrac{x}{y} + \dfrac{y}{x}$ and $\dfrac{x}{y}$.

If $a : b = c : d$, prove:

7. $a + b : a - b = c + d : c - d$.

8. $a : b = \sqrt{a^2 + c^2} : \sqrt{b^2 + d^2}$.

9. $\sqrt{a^2 + b^2} : \sqrt{c^2 + d^2} = \sqrt[3]{a^3 + b^3} : \sqrt[3]{c^3 + d^3}$.

10. $a + b : c + d = \sqrt{2a^2 - 3b^2} : \sqrt{2c^2 - 3d^2}$.

11. $a^2 + ab + b^2 : c^2 + cd + d^2 = a^2 - ab + b^2 : c^2 - cd + d^2$.

12. $a^2 + 3ab + b^2 : c^2 + 3cd + d^2 = 2ab + 3b^2 : 2cd + 3d^2$.

If $a : b = b : c$, prove:

13. $a + b : b + c = \sqrt{a} : \sqrt{c}$.

14. $a^2 + ab + b^2 : b^2 + bc + c^2 = a : c$.

15. $a^3 + c^3 : a^2 - b^2 + c^2 = a + c : 1$.

If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, prove:

16. $\frac{x + y}{a + b} = \frac{y + z}{b + c} = \frac{z + x}{c + a}$.

17. $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$.

From each of the following prove $a : b = c : d$.

18. $(a + b - 3c - 3d)(2a - 2b - c + d) = (2a + 2b - c - d)(a - b - 3c + 3d)$.

19. $(3a + 6b + c + 2d)(3a - 6b - c + 2d) = (3a - 6b + c - 2d)(3a + 6b - c - 2d)$.

Solve the following equations, using as many theorems in proportion as possible:

20. $\begin{cases} x^3 + y^3 : x^3 - y^3 = 70 : 38. \\ xy = 6. \end{cases}$

21. $\begin{cases} x^3 - y^3 : (x - y)^3 = 38 : 2. \\ x + y = 5. \end{cases}$

22. $\begin{cases} x^3 + y^3 : (x + y)^3 = 21 : 81. \\ 2x^2y + 2xy^2 = 360. \end{cases}$

23. $\frac{3x^4 + x^2 - 2x - 3}{3x^4 - x^2 + 2x + 3} = \frac{5x^4 + 2x^2 - 7x + 3}{5x^4 - 2x^2 + 7x - 3}$.

24. $\frac{\sqrt{x + a} + \sqrt{x - a}}{\sqrt{x + a} - \sqrt{x - a}} = \frac{m + n}{m - n}$.

25. $\frac{\sqrt{2a^2 - x^2} + b\sqrt{2a - x}}{\sqrt{2a^2 - x^2} - b\sqrt{2a - x}} = \frac{\sqrt{a} + b}{\sqrt{a} - b}$.

Note. — Equations 20, 24, 25, § 149; 6, 7, 9, § 154; and 20, 21, 23, § 160, may be readily reduced to a form convenient for solution by the principle of Composition and Division.

XXII. VARIATION.

DEFINITIONS.

190. A **Variable** is a quantity which may have different values in the same expression (§ 24).

191. A **Constant** is a quantity whose value is fixed in any given expression.

Thus, in the formula for the area of a triangle, *area* = $\frac{1}{2}ba$, or $A = mab$, A, a, b, are variables, and m is a constant.

192. EXERCISES.

1. Express in symbolic form each of the following: x varies as y; x varies inversely as y; x varies jointly as y and z; x varies directly as y and inversely as z.

2. Express each form in Exercise 1 as an equation.

3. If $x \propto y$, and equals 8 when $y = 15$, what is its value when $y = 10$?

4. If $x \propto \frac{1}{y}$, and equals 15 when $y = 4$, what is its value when $y = 10$?

5. If $x \propto yz$, and $y = 5$ when $x = 9$ and $z = 7$, what is the value of y when $z = 70$ and $x = 54$?

6. If $x \propto \frac{y}{z}$, and $y = 10$ when $x = 14$ and $z = 14$, what is the value of z when $x = 49$ and $y = 45$?

7. If $x \propto y$, prove that $x^n \propto y^n$; also that $x^2 + y^2 \propto x^2 - y^2$.

8. If $x \propto y$ and $y \propto z$, prove that $x \pm y$ and $\sqrt{xy}$ each vary as z.

9. If $A \propto BC$, $B \propto D^2$, and $C \propto \frac{1}{A}$, prove that A varies as D.

10. If x varies directly as $\sqrt{y}$ and inversely as z^3, and $x = 3$ when $y = 256$ and $z = 2$, what is the value of y when $x = 24$ and $z = \frac{1}{2}$?

11. x varies as the sum of two quantities, one of which is constant and the other varies as xy; and when $x = 3$, $y = 1$; also when $x = 9$, $y = 4$. Find the value of x in terms of y.

12. x varies as the difference of two quantities, one of which is constant and the other varies as y^2; and when $x = 3$, $y = 2$; also when $x = 18$, $y = 1$. Find the value of x in terms of y.

13. x varies as the sum of two quantities, one of which varies as y and the other as y^2. If $x = 7$ when $y = 1$, and $= 39$ when $y = 3$, what is the value of x in terms of y?

14. How many acres can 16 men reap in 3 days, if 7 men can reap 126 acres in 20 days?

15. How many bushels of grain will 30 horses eat in 16 days, if 12 horses eat 48 bushels in 20 days?

16. The area of a circle whose radius is 10 feet is 314.159 square feet. Find the area of a circle whose radius is 12 feet, from the law that the area of a circle varies as the square of its radius.

17. From the same law show that the area of a circle $2\frac{1}{2}$ inches in diameter is equal to the sum of the areas of two circles $1\frac{1}{2}$ and 2 inches in diameter.

XXIII. ARITHMETICAL PROGRESSION.

DEFINITIONS.

193. A **Series** is a set of numbers or quantities which succeed each other by a fixed law.

194. An **Arithmetical Progression** is a series in which each term is derived from the one next preceding it, by adding to that term a constant quantity called the *common difference.*

195. QUESTIONS AND EXERCISES.

1. By means of what two formulas can all questions in Arithmetical Progression be solved?

2. Show how these two formulas are obtained.

Find l and S in each of the following:

3. $4, 5\frac{1}{4}, 6\frac{1}{2}, \ldots$ to 13 terms.
4. $20, 16\frac{1}{4}, 12\frac{1}{2}, \ldots$ to 101 terms.
5. $1, -\frac{1}{5}, -\frac{7}{5}, \ldots$ to 11 terms.

Which term of the series:

6. $5, 8, 11, \ldots$ is 65?
7. $4, 11, 18, \ldots$ is 312?
8. $\frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \ldots$ is 18?

How many terms must be taken of the series:

9. $42, 39, 36, \ldots$ to make 315?
10. $-16, -15, -14, \ldots$ to make 74?
11. $-10\frac{1}{2}, -9, -7\frac{1}{2}, \ldots$ to make -42?

Find the first three terms:

12. When the 27th term is 186, and the 45th 312.
13. When the 15th term is 25, and the 29th 46.
14. When the 16th term is 214, and the 51st 739.

15. Insert 8 arithmetical means between 1 and 0.
16. Insert 22 arithmetical means between 8 and 54.
17. Find the arithmetical mean between $7\frac{3}{4}$ and $-23\frac{1}{4}$.
18. Find the arithmetical mean between a and

$$\frac{an + 2b - a}{n + 1}.$$

19. Given $a = 7$, $d = 2$, $S = 1927$; find l and n.
20. Given $a = 10$, $n = 5$, $S = -20$; find l and d.
21. Given $d = \frac{3}{4}$, $l = 41$, $S = 1127$; find n and a.

22. The forms x, $x+y$, $x+2y$, etc.; $x-3y$, $x-y$, $x+y$, $x+3y$; $x-2y$, $x-y$, $x, x+y$, $x+2y$, are sometimes used to represent the terms of a series. What is the common difference in each, and when would one form be preferable to another?

23. The sum of three numbers in A. P. is 27, and their product is 504. What are the numbers?

24. The sum of four numbers in A. P. is 20, and the sum of their cubes is 800. What are the numbers?

25. The sum of five numbers in A. P. is 75, and the product of the greatest and least is 161. What are the numbers?

26. The sum of three numbers in A. P. is 12, and the sum of their squares is 66. What are the numbers?

27. A sets out from a place and travels uniformly $2\frac{1}{2}$ miles per hour. B sets out from the same place 3 hours afterwards and follows A at the rate of 3 miles the first hour, increasing his rate one half a mile each hour. In how many hours will he overtake A?

28. A man agrees to pay a debt of \$3600 in 40 annual payments which form an Arithmetical Progression. When 30 of these payments are made he dies, leaving one-third of the debt unpaid. What was the first payment?

29. Sum the series 1, 3, 5, 7, . . . to $2n$ terms.

30. The sum of n terms of a series is n^2. Find the first term and common difference, the last term being $2n-1$.

31. If a, b, c, and d are in A. P., prove that $bc-ad$ is positive.

Find the sum of n terms of the series:

32. $\frac{2}{3}, \frac{3}{4}, \ldots$

33. $(a-b), (3a-2b), \ldots$

34. $\frac{a-1}{a}, \frac{a-2}{a}, \frac{a-3}{a}, \ldots$

35. Find the sum of $n + 2$ terms of the series

$$a, \frac{na+c}{n+1}, \frac{(n-1)a+2c}{n+1}, \ldots$$

36. If a^2, b^2, c^2 are in A. P., prove that

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A. P.}$$

DERIVATION OF FORMULAS.

196. The following formulas, selected from a possible twenty, are to be derived from the fundamental formulas $l = a + (n-1)d$, and $S = \left(\frac{a+l}{2}\right)n$. They have no especial value in solving problems and need not be committed to memory. Their derivation is introduced as an excellent exercise in the reduction of literal equations; in the changing of forms, given or obtained, to required forms; and in using any letter as the unknown quantity.

No.	Given.	Required.	Formulas.
1.	$d\,n\,S$	l	$l = \frac{S}{n} + \frac{(n-1)d}{2}.$
2.	$d\,n\,l$	S	$S = \frac{1}{2}n[2l - (n-1)d].$
3.	$a\,d\,l$	S	$S = \frac{l+a}{2} + \frac{l^2-a^2}{2d}.$
4.	$a\,n\,S$	d	$d = \frac{2(S-an)}{n(n-1)}.$
5.	$a\,l\,S$	d	$d = \frac{l^2-a^2}{2S-l-a}.$
6.	$n\,l\,S$	d	$d = \frac{2(nl-S)}{n(n-1)}.$
7.	$a\,d\,S$	l	$l = -\frac{1}{2}d \pm \sqrt{2dS + (a-\frac{1}{2}d)^2}.$
8.	$d\,l\,S$	a	$a = \frac{1}{2}d \pm \sqrt{(l+\frac{1}{2}d)^2 - 2dS}.$
9.	$a\,d\,S$	n	$n = \frac{d-2a \pm \sqrt{(d-2a)^2+8dS}}{2d}.$
10.	$d\,l\,S$	n	$n = \frac{2l+d \pm \sqrt{(2l+d)^2-8dS}}{2d}.$

XXIV. GEOMETRICAL PROGRESSION.

197. Definition. — A **Geometrical Progression** is a series in which each term is derived from the one next preceding it, by multiplying that term by a constant quantity called the *ratio.*

198. QUESTIONS AND EXERCISES.

1. By means of what two formulas can all questions in Geometrical Progression be solved?

2. Show how these two formulas are obtained, and how the formula for S is made applicable to an infinite decreasing series.

Find the required terms in each of the following:

3. The 5th and 8th in the series 3, 6, 12, . . .

4. The 7th and 14th in the series $\frac{1}{64}$, $\frac{1}{32}$, $\frac{1}{16}$, . . .

5. The 6th and 9th in the series .008, .04, .2, . . .

6. The 50th in the series $2x$, 1, $\dfrac{1}{2x}$, . . .

7. The nth term in the series x, x^4, x^7, . . .

8. The $2n$th term in the series 3, -3^2, 3^3, . . .

Find the sum of:

9. Five terms of the series 9, 3, 1, . . .

10. Eight terms of the series $1\frac{1}{2}$, $1\frac{1}{3}$, $1\frac{5}{27}$, . . .

11. n terms of the series 8, 12, 18, . . .

Find n in each of the series when:

12. $l = 1024$, $a = 8$, $S = 2040$.

13. $l = \frac{1}{243}$, $a = 3$, $r = -\frac{1}{3}$.

14. $S = \frac{369}{160}$, $a = \frac{2}{5}$, $r = \frac{5}{4}$.

Find the required letters in the following:

15. Given $a = 10$, $r = 2$, $S = 2550$; find n and l.

16. Given $l = 243\sqrt{3}$, $r = \sqrt{3}$, $S = 364(\sqrt{3} + 1)$; find a and n.

17. Given $a = 2$, $n = 10$, $l = 1024$; find r and S.

18. Given $a = \frac{2}{3}$, $r = -\sqrt{3}$, $n = 8$; find S and l.

19. Given $a = \dfrac{1}{\sqrt{2}}$, $l = \dfrac{512}{\sqrt{2}}$, $S = \dfrac{1023}{\sqrt{2}}$; find n and r.

Find the sum to infinity of:

20. $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots$ **21.** $\frac{4}{3}, 1, \frac{3}{4}, \ldots$

22. $1, -\frac{1}{2}, \frac{1}{4}, \ldots$

23. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}, \dfrac{1}{2-\sqrt{2}}, \dfrac{1}{2}, \ldots$

24. $\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}+1}, \dfrac{2}{4+3\sqrt{2}}, \ldots$

25. .151515 . . . **27.** .42323 . . .

26. .281313 . . . **28.** .123123 . . .

29. 32.020352035 . . .

30. Find the geometrical mean between 12 and 108.

31. Find the geometrical mean between $4x^2 - 12x + 9$ and $9x^2 + 12x + 4$.

32. Insert five geometrical means between 1 and $\frac{8}{729}$.

33. If the arithmetical mean between two numbers is $1 + a^2$ and the geometrical mean $1 - a^2$, what are the numbers?

34. If a, b, c, and d are in Geometrical Progression, prove that $(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$.

35. x, xy, xy^2, etc.; $x, \sqrt{xy}, y$; $\dfrac{x^2}{y}, x, y, \dfrac{y^2}{x}$, are forms used in solving questions in Geometrical Progression. What is the ratio in each, and when would one be preferred to another?

36. The sum of three terms in Geometrical Progression is 63, and the third is 45 greater than the first. What are the terms?

37. The sum of the first 8 terms of a Geometrical Progression is 3280, and the sum of the first 4 terms is 40. What are the first three?

38. The sum of three terms of a Geometrical Progression is 21, and the sum of their squares is 189. What are the terms?

39. The continued product of four numbers in Geometrical Progression is 2916, and the sum of the means is 15. What are the numbers?

199. DERIVATION OF FORMULAS.

From the formulas $l = ar^{n-1}$ and $S = \frac{lr - a}{r - 1}$, derive each of the following formulas:

No.	Given.	Required.	Formulas.
1.	$a\ r\ S$	l	$l = \frac{a + (r-1)S}{r}$.
2.	$r\ n\ S$	l	$l = \frac{(r-1)Sr^{n-1}}{r^n - 1}$.
3.	$a\ r\ n$	S	$S = \frac{a(r^n - 1)}{r - 1}$.
4.	$a\ n\ l$	S	$S = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$.
5.	$r\ n\ l$	S	$S = \frac{(r^n - 1)l}{(r-1)r^{n-1}}$.
6.	$r\ n\ S$	a	$a = \frac{(r-1)S}{r^n - 1}$.
7.	$r\ l\ S$	a	$a = rl - (r-1)S$.
8.	$n\ l\ S$	a	$a(S-a)^{n-1} - l(S-l)^{n-1} = 0$.
9.	$a\ n\ S$	r	$r^n - \frac{S}{a}r + \frac{S-a}{a} = 0$.
10.	$a\ l\ S$	r	$r = \frac{S-a}{S-l}$.

XXV. HARMONICAL PROGRESSION.

200. Definition. — A series of numbers or quantities is said to be in **Harmonical Progression** when their reciprocals are in arithmetical progression.

201. There is no general formula for the sum of any number of terms of a Harmonical series. Questions in Harmonical Progression are generally solved by inverting the terms and using the formulas of arithmetical progression, and inverting the results.

EXERCISES.

1. The first two terms of an H. P. are 4 and $\frac{8}{5}$, what is the 9th?

2. The first two terms of an H. P. are 3 and $\frac{3}{2}$, what is the nth?

3. Continue the H. P. $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{3}$, three terms each way.

4. Insert 4 harmonical means between 1 and 6.

5. Insert 1 harmonical mean between

$$\frac{1}{(x+y)^2} \text{ and } \frac{1}{(x-y)^2}.$$

6. The sum of two contiguous terms in H. P. is $\frac{16}{63}$ and their product $\frac{1}{63}$, find 4 terms of the series.

If a, b, and c, are in Harmonical Progression, prove:

7. $a : c = a - b : b - c$.

8. b, the harmonical mean, equal to $\dfrac{2ac}{a+c}$.

9. a, $a - c$, $a - b$, in H. P.

10. c, $c - a$, $c - b$, in H. P.

11. $\dfrac{b+a}{b-a} + \dfrac{b+c}{b-c} = 2$.

12. $(b + c - a)^2$, $(c + a - b)^2$, $(a + b - c)^2$, in Arithmetical Progression.

13. $a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$, in Geometrical Progression.

14. If the arithmetical mean between two quantities is 1, prove that the harmonical mean is the square of the geometrical mean.

XXVI. PERMUTATIONS AND COMBINATIONS.

DEFINITIONS.

202. The different orders in which a number of things can be arranged are called their **Permutations**.

Thus, the permutations of *a, b, c,* are *abc, acb, bca, bac, cab, cba.*

203. The different groups which can be made out of a number of things without regard to their order are called the **Combinations** of the things.

Thus, the combinations of *a, b, c,* taken two at a time, are *ab, bc, ca.*

204. In making up the permutations or combinations of a number of things, any number of them can be taken at a time.

205. QUESTIONS AND EXERCISES.

What is the formula for:

1. The permutations of n different things taken r at a time?

2. The permutations of n things, of which some are alike, taken all at a time?

3. The combinations of n different things taken r at a time?

What is the number of:

4. The permutations of 8 things taken 7 at a time?

5. The permutations of 25 things taken 5 at a time?

6. The combinations of 24 things taken 4 at a time?

7. The combinations of 19 things taken 14 at a time?

8. How many different permutations can be formed from the letters of the word Cincinnati? from Mississippi?

9. The number of permutations of n things taken 3 at a time is six times the number taken 2 at a time: find the value of n.

10. The number of permutations of x things taken 4 at a time is 15 times the number of permutations of $x - 2$ things taken 3 at a time: find the value of x.

11. How many words, each containing 3 consonants and 2 vowels, can be formed from 6 consonants and 4 vowels?

12. How many committees, each containing 3 Republicans and 4 Democrats, can be made up from 12 Republicans and 16 Democrats?

13. How many different numbers of 5 figures each can be formed from the 10 digits?

14. The number of permutations of n things taken all at a time is 5040: find the value of n.

15. Out of 8 letters, x of which are alike, 336 words can be formed: find the value of x.

16. How many words can be formed from 6 letters taken all at a time, three of which cannot be separated? How many if the three are allowed only one order?

17. In how many ways can 6 men be chosen from 10? In how many of these combinations would a particular man be found?

18. In how many ways can 2 sixes, 3 fours, and 3 fives, be thrown with 8 dice?

19. In how many ways can 3 persons be seated at a round table? In how many, 6? In how many, 7?

XXVII. UNDETERMINED COEFFICIENTS.

206. State and prove the *Theorem of Undetermined Coefficients.*

APPLICATIONS OF THE THEOREM.

207. DIVISION, OR THE EXPANSION OF FRACTIONS INTO SERIES.

Expand each of the following to five terms:

1. $\frac{1+2x}{1-x-x^2}$.
2. $\frac{3+x}{2-x-x^2}$.
3. $\frac{1+x+x^2}{1-x-x^2}$.
4. $\frac{2-3x}{1+x+x^2}$.
5. $\frac{1-x}{1+x-2x^2}$.
6. $\frac{1}{1+x+2x^2}$.
7. $\frac{1+2x^2+x^4}{1-x^2-x^4}$.
8. $\frac{2+x+3x^2}{2-x-x^2}$.
9. $\frac{2-x^2}{1+x^2+x^4}$.
10. $\frac{x^2+x^3+x^4}{1-x-x^3}$.
11. $\frac{x}{1-2x}$.
12. $\frac{x+x^2+x^3}{1-x-x^2+x^3}$.
13. $\frac{3}{x^2+x^3}$.
14. $\frac{1+x}{x^2+2x^3+2x^4}$.
15. $\frac{3+x^2}{x^2+2x^3}$.
16. $\frac{1+x+x^2}{x-x^2}$.
17. $\frac{2+x^2+x^3}{2x+x^2+x^3}$.
18. $\frac{1}{x^2-2x^3+x^4}$.

208. EXTRACTING ROOTS, OR THE EXPANSION OF RADICALS INTO SERIES.

Expand each of the following to five terms:

1. $\sqrt{1+3x}$.
2. $\sqrt{x^2+2x^3+3x^4}$.
3. $\sqrt{1+x+x^2}$.
4. $\sqrt[3]{1+3x+x^2}$.
5. $\sqrt[3]{x^3+5x^4+x^5}$.
6. $\sqrt[3]{8x^3+x^4+2x^5}$.

DECOMPOSITION OF FRACTIONS.

Case I.

209. When the denominator can be resolved into unequal factors of the first degree.

Resolve each of the following into partial fractions:

1. $\dfrac{3x}{x^2-7x+6}$.
2. $\dfrac{x+2}{6x^2+13x+6}$.
3. $\dfrac{2}{x(x+2)(x+3)}$.
4. $\dfrac{x^2-10x+13}{(x-1)(x-2)(x-3)}$.
5. $\dfrac{2x+1}{(x+1)(x^2+5x+6)}$.
6.* $\dfrac{2x^2+3x+1}{(1-2x)(1-x)}$.
7. $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}$.
8. $\dfrac{2x^2+x+1}{2x^2+x-3}$.
9. $\dfrac{2x^3+x^2-x-3}{x(x+1)(2x+3)}$.

Case II.

210. When the denominator can be resolved into equal factors of the first degree.

Resolve each of the following into partial fractions:

1. $\dfrac{2x+3}{x^2+10x+25}$.
2. $\dfrac{x-1}{4x^2+12x+9}$.
3. $\dfrac{12+x}{x^2+6x+9}$.
4. $\dfrac{x^2+3x+1}{(x+1)^3}$.
5. $\dfrac{4x+1}{(x+5)^4}$.
6. $\dfrac{2x^2+3x+1}{x^4}$.

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* Reduce to a mixed form and decompose the fractional part whenever the degree of the numerator equals or exceeds that of the denominator.

CASE III.

211. When the denominator can be resolved into factors of the first degree, some of which are equal.

Resolve each of the following into partial fractions:

1. $\dfrac{8 - x}{(1 + x)(2 - x)^2}$.
2. $\dfrac{5 - 9x}{(1 + x)(1 - 3x)^2}$.
3. $\dfrac{1 + 7x - x^2}{(1 - 10x)(1 + 3x)^2}$.
4. $\dfrac{x^2}{(x + 1)^2(x + 2)^2}$.
5. $\dfrac{x^4 - 3x^3 - 3x^2 + 10}{(x - 3)(x + 1)^2}$.
6. $\dfrac{1 + 2x}{x^2(x - 1)(x + 2)^3}$.

CASE IV.

212. When the denominator can be resolved into factors, equal or unequal, and one or more of them of the second degree.

Resolve each of the following into partial fractions:

1. $\dfrac{7 + x}{(1 + x)(1 + x^2)}$.
2. $\dfrac{x^3 - 3}{(x + 2)(x^2 + 1)}$.
3. $\dfrac{26x^2 + 208x}{(x^2 + 1)(x + 5)}$.
4. $\dfrac{x^2 + x}{(x^2 + 4)(x - 1)^2}$.
5. $\dfrac{x^3 + 2}{(x - 1)^2(x^2 + 1)}$.
6. $\dfrac{x^2 - x + 1}{(x^2 + 2)(x^2 + x + 1)}$.

213. REVERSION OF SERIES.

To revert a series $x = Ay + By^2 + Cy^3 + \dots$ is to express the value of y in terms of x; as, $y = A'x + B'x^2 + C'x^3 + \dots$

Revert each of the following series to four terms:

1. $y = x + x^2 + 2x^3 - 2x^4 + 2x^5 - \dots$
2. $y = x + 4x^2 + 4x^3 + 4x^4 + 4x^5 + \dots$
3. $y = x + x^2 - \dfrac{x^3}{2} + \dfrac{x^4}{4} - \dfrac{x^5}{8} - \dots$

4. $y = 2 + x + 3x^2 - 3x^3 + 3x^4 - \dots$
5. $y = 1 + x + 2x^2 - 2x^3 + 2x^4 - \dots$
6. $y = x - 2x^3 + 2x^5 + 3x^7 + 4x^9 - \dots$
7. $y = x^2 + 2x^4 + 2x^6 - x^8 + x^{10} - \dots$

XXVIII. CONTINUED FRACTIONS.

DEFINITIONS.

214. A **Continued Fraction** is a fraction in the form of

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}}, \text{ also written } \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4}.$$

215. The simple fractions $\frac{1}{2}$, $\frac{1}{3}$, etc., are called *elements* of the continued fraction; and a continued fraction is *terminating* or *infinite* according as its elements are finite or infinite in number.

216. When the elements continually recur in the same order, it is called a *periodic* continued fraction; and a periodic continued fraction is *simple* or *mixed* according as the repetitions commence at the beginning or not.

217. The **Convergents,** or **Approximate Values,** of the continued fraction, are the fractions obtained by terminating the continued fraction with each element and simplifying the resulting complex fraction.

218. EXERCISES.

Reduce the following to continued fractions:

1. $\frac{102}{19}$. 3. $\frac{13}{16}$. 5. $\frac{119}{225}$. 7. .375.

2. $\frac{23}{27}$. 4. $\frac{25}{26}$. 6. $\frac{37}{100}$. 8. .028.

Find the simple fractions equivalent to:

9. $\frac{1}{6+}\,\frac{1}{2+}\,\frac{1}{2}$.

10. $\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{16}$.

11. $1+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4}$.

12. $\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5+}\,\frac{1}{6}$.

13. $\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{5}$.

14. $\frac{1}{4+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2}$.

15. $\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2}$.

16. $\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3}$.

Reduce to the form of continued fractions and find the convergents of:

17. $\frac{12}{19}$. **18.** $\frac{31}{37}$. **19.** $\frac{117}{129}$. **20.** $\frac{52}{61}$. **21.** $\frac{9}{17}$.

Change to the form of a continued fraction, and find the sixth convergent in each of the following:

22. $\sqrt{8}$. **23.** $\sqrt{2}$. **24.** $\sqrt{13}$. **25.** $\sqrt{22}$. **26.** $2\sqrt{3}$.

Reduce each of the following to an equivalent radical:

27. $1+\frac{1}{2+}\,\frac{1}{2+}\cdots$

28. $3+\frac{1}{6+}\,\frac{1}{6+}\cdots$

29. $5+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{10+}\cdots$

30. $\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3+}\cdots$

Express as a continued fraction the positive root of:

31. $x^2-4x-3=0$.

32. $x^2+2x-1=0$.

Find the limits of the error in taking:

33. $\frac{33}{8}$ for $\sqrt{17}$. **34.** $\frac{61}{13}$ for $\sqrt{22}$.

COLLEGE AND SCIENTIFIC SCHOOL ENTRANCE EXAMINATION PAPERS IN ALGEBRA.

HARVARD UNIVERSITY.

ELEMENTARY.

JUNE, 1890.

1. Find the greatest common divisor of $4x^4 - 6x^3 + 9x^2 - 5x + 3$ and $8x^3 + 8x^2 + 9$.

2. Solve the equation

$$\sqrt{2x + 9} + \sqrt{3x - 15} = \sqrt{7x + 8}.$$

3. If $\dfrac{x}{y + z} = a$, $\dfrac{y}{x + z} = b$, $\dfrac{z}{x + y} = c$, find the value of

$$\frac{a}{1 + a} + \frac{b}{1 + b} + \frac{c}{1 + c}.$$

4. A and B start at the same time from two towns, and travel towards each other. When they meet, B has travelled a miles more than A; it will take A b days longer to reach the town B has left, and B c days longer to reach the town A has left. Find the distance between the towns.

5. If $a : b = c : d$, prove $a : b = \sqrt{3a^2 + 5c^2} : \sqrt{3b^2 + 5d^2}$.

JUNE, 1891.

1. Solve the equation

$$\frac{2}{\frac{1}{b}\left(\frac{x}{2a} - 2\right)} - \frac{b}{a}\left[\frac{4a}{\frac{4b}{x} - \frac{b}{a}} - \frac{1}{\frac{1}{x} - \frac{1}{b}}\right] = 0.$$

2. Two tanks, A and B, are discharging water; A at the rate of x barrels per hour, and B at the rate of $(x + 100)$

barrels per hour. At a time $(1+y)$ hours after noon, A contains 480 barrels less than at noon; and at a time $(1-y)$ hours after noon, B contains 400 barrels less than at noon. Find the rate at which each tank is discharging water; and the times $(1+y)$ and $(1-y)$ hours after noon.

Obtain two sets of answers, and interpret negative results.

3. Find the Greatest Common Measure and the Least Common Multiple of $12x^2-29x+14$ and $18x^2+3x-10$.

4. Find the fourth term of $\left(\sqrt{a}-\frac{b^7}{6a^5}\right)^{81}$.

5. Find a mean proportional between $4x^3-3x-1$ and $x-1$.

JUNE, 1892

1. Solve the equation

$$\frac{2a(a+b)-b^2x}{bx-2a}=\frac{2}{\frac{b}{a^2}\left(\frac{1}{x}-\frac{b}{2a}\right)}-b\left(1-\frac{b}{a-\frac{b}{2x}}\right);$$

and reduce the answers to their lowest terms.

2. Find a mean proportional between

$$2x-3y \text{ and } 18x^3-15x^2y-16xy^2-3y^3.$$

Find the values of these quantities and of the mean proportional, when $x=1$ and $y=-1$.

What *two values* does the mean proportional admit?

3. Find the Greatest Common Measure and the Least Common Multiple of $6x^3+7x^2-5x$ and $15x^4+31x^3+10x^2$.

4. A certain railway runs due east and west, P, Q, and R being successive stations on the road from east to west, and so situated that PQ = 216 miles, PR = 240 miles.

Two trains, numbered 1 and 2, on parallel tracks, pass P simultaneously at 12 o'clock noon. The rate of motion of train No. 2 from east to west is 8 miles per hour less than that of train No. 1; and train No. 2 passes Q $1\frac{1}{2}$ hours later than train No. 1 passes R.

Find the rate and direction of motion of each train; and find the hour at which train No. 1 passes R, and the hour at which train No. 2 passes Q.

(Obtain two sets of answers, and interpret negative results.)

5. Find the prime factors of the numerical coefficient of the 10th term of $(x-y)^{29}$. Denoting that coefficient by C, find the 10th term of $\left(\frac{2a^2}{b^3}-\sqrt[5]{b^2}\right)^{29}$, in its most reduced form.

JUNE, 1893.

1. Two men, A and B, had a money-box containing $210, from which each drew a certain sum daily; this sum being fixed for each, but different for the two. After six weeks, the box was empty. Find the sum which each man drew daily from the box; knowing that A *alone* would have emptied it five weeks earlier than B *alone*.

Obtain *two solutions*, and interpret the *negative answer*.

2. Solve the equation

$$\frac{x+b}{2a}+\frac{2a}{x-b}=1-\frac{2a}{b}\left(1-\frac{2a-b}{x-b}\right).$$

3. Reduce to their lowest common denominator

$$\frac{1}{12x^3-2x^2-20x-6} \text{ and } \frac{1}{4x^3-6x^2-4x+6}$$

and find, and reduce to their lowest terms, the *difference* and the *quotient* of these two fractions.

4. Write out $(x-y)^9$.

Find the 8th term of $\left(\frac{4\sqrt[4]{b^8}}{3a^5}-\frac{1}{2}a\sqrt{a}\,.\,b^{-\frac{1}{5}}\right)^9$; reducing the answer to the simplest possible form, in which it is free from negative and fractional exponents, and has only one radical sign.

YALE COLLEGE.

SEPTEMBER, 1891.

1. (*a*) Find product of
$(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)$.
(*b*) Divide $(x+y)+z$ by $(x+y)^{\frac{1}{3}}+z^{\frac{1}{3}}$.

2. Factor
(*a*) $9x^2+9x+2$. (*b*) $(x^3+3x)^2-(3x^2+1)^2$.

3. Solve the equations

$$(a)\quad \sqrt{a-x}+\sqrt{b-x}+\frac{b}{\sqrt{b-x}}=0.$$

$$(b)\quad 5x^2+7x-160=0.$$

4. Find values of x and y in the simultaneous equations

$$x+y=2a,$$
$$x^2+y^2=2a^2.$$

5. Solve the equation $\left(x+\frac{1}{x}\right)^2+4\left(x+\frac{1}{x}\right)=12$.

6. Change $\dfrac{1}{1+\sqrt{-2}+\sqrt{-3}}$ into a fraction having a rational denominator.

7. In how many ways could a base-ball nine be chosen out of twelve players?

In how many ways could this be done if only one of the twelve could be used as pitcher while each of the others could play in any position?

8. Write the middle term of the series $(1-2x^2)^8$

SEPTEMBER, 1892.

1. Find four factors of $8ax^2-24axy+18ay^2$, and also of

$$(a^2+a-4)^2-4.$$

2. Divide $a^{-\frac{4}{3}}+a^{-\frac{2}{3}}b^2+b^4$ by $a^{-\frac{2}{3}}+a^{-\frac{1}{3}}b+b^2$.

3. There are two numbers such that one is as much greater than 19 as the other is less, and their difference is to their sum as 3 is to 19. Find the numbers.

4. Square $a - b - 2\sqrt{-ab}$.

5. Solve the equation $x^{\frac{6}{5}} - 10x^{\frac{3}{5}} = -24$.

6. Solve $\dfrac{1}{x + \sqrt{2 - x^2}} + \dfrac{1}{x - \sqrt{2 - x^2}} = \dfrac{x}{2}$.

7. Find x and y from the equations $x^2 + 3xy = -14$,
$$xy + 4y^2 = 30.$$

8. Find the tenth term of the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

9. Write the first four terms of $(x^3 - 2ay)^{10}$.

JUNE, 1893.

1. Write the factors of the following expressions:
$$x^4 - (x - 6)^2, \text{ and } m^6 - 64n^6.$$

2. Simplify
$$\frac{x - 1 - \dfrac{12}{x + 4}}{x - 5 + \dfrac{12}{x + 3}}.$$

3. A and B can do a piece of work in m days; B and C can do it in n days; C and D in p days; and D and A in r days. In how many days can all working together do it?

4. Multiply $x + y\sqrt{-z}$ by $y - z\sqrt{-x}$.

5. Solve the equation $y^2 + 2(a + 6)y = -18a$.

6. Extract the square root of
$$x^3y^{-\frac{2}{3}} - 4x^{\frac{3}{2}}y^{-\frac{1}{3}} + 6 - 4x^{-\frac{3}{2}}y^{\frac{1}{3}} + x^{-3}y^{\frac{2}{3}}.$$

7. Simplify
$$\frac{1}{1 + \dfrac{1}{1 + \dfrac{1}{a}}}.$$

8. Write the 6th term of $(a - 2b)^{\frac{1}{2}}$.

CORNELL UNIVERSITY.

1892.

ELEMENTARY.

1. Resolve the following expressions into their prime factors:

$x^6 - y^6$; $-b^3 + a^3 - 3ab(a - b)$; $2x^3 + 4x^2 - 70x$; $5x^2 - 8xy + 3y - 5x + 3y^2$.

2. Simplify

$$\left\{x - \frac{\frac{1}{y^{-1}}(x - y)}{x + y}\right\} : \left\{1 - \frac{xy - y^2}{x^2}\right\} \cdot \left\{x - \frac{\frac{1}{y^{-2}}(x - y)}{x^2 + y^2}\right\}.$$

3. Find the *L. C. M.* of

$(a + b)^2 - c^2$, $a^2 - (b + c)^2$, $(a + c)^2 - b^2$;

and of $6x^3 - 11x^2y + 2y^3$, $9x^3 - 22xy^2 - 8y^3$.

4. Prove that $(a^m)^n = a^{mn} = (a^n)^m$ when m, n are positive or negative, integral or fractional.

5. Two workmen together complete some work in 20 days; if the first had worked twice as fast, and the second half as fast, they would have completed it in 15 days; how long would it have taken each alone to perform the same work?

6. From an 80-gallon cask, filled with wine, a person draws off a certain number of gallons, which he replaces by water. From this mixture he again draws off the same number of gallons as before, and again replaces it by water. It is now found that 16 gallons of the mixture in the cask contains only 9 gallons of wine. How many gallons did he draw off each time?

7. Find x, y from the simultaneous equations

$2x^2 - 3xy + y^2 = 24$, $3x^2 - 5xy + 2y^2 = 33$.

ADVANCED.

1. Given log 2 = 0.30103, log 3 = 0.47712; find the logarithm of $\dfrac{2\sqrt[3]{5}}{3^{\frac{5}{6}}}$.

2. Extract the square root of $7 - 30\sqrt{-2}$.

3. Find the number of *selections*, and also the number of *arrangements*, that can be made by taking 4 letters of the word *mathematics*.

4. By the method of undetermined coefficients resolve

$$\frac{x^4 - 3x^3 - 3x^2 + 10}{(x-3)(x+1)^2}$$

into partial fractions; and expand one of these partial fractions into a series, writing the first 4 terms.

5. When is an infinite series said to be *convergent? divergent?* Show whether $\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \frac{x^3}{5\cdot 6} + \frac{x^4}{7\cdot 8} + \ldots$ is convergent or divergent. Examine the three cases; $x < 1$, $x = 1$, and $x > 1$.

6. Define a *root* of an equation. Knowing that the roots of $54x^3 - 39x^2 - 26x + 16 = 0$ are in geometric progression, solve the equation.

1893.

ELEMENTARY.

1. Resolve into factors

(*i*) $a^2 + x^2 - (y^2 + z^2) - 2(yz - ax)$.
(*ii*) $a^2 - 9b^2 + a + 3b$.

2. Simplify

$$\frac{\frac{1}{x} - \frac{x+a}{x^2+a^2}}{\frac{1}{a} - \frac{a+x}{a^2+x^2}} + \frac{\frac{1}{x} - \frac{x-a}{x^2+a^2}}{\frac{1}{a} - \frac{a-x}{a^2+x^2}}.$$

3. Simplify as much as possible the expression

$$\left(a^{-\frac{1}{2}}x^{\frac{1}{3}}\sqrt{ax^{-\frac{1}{3}}\sqrt[4]{x^{\frac{4}{3}}}}\right)^{\frac{1}{3}}.$$

4. Express in the simplest form

(*i*) $2\sqrt[3]{189} + 3\sqrt[3]{875} - 7\sqrt[3]{56}$.
(*ii*) $3\sqrt{147} - \frac{7}{3}\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{27}}$.

5. Solve the equation $\frac{6x+8}{2x+1} - \frac{2x+38}{x+12} = 1$.

6. Solve the simultaneous equations

$$\frac{34}{x^2+y^2} = \frac{15}{xy},\ x+y=8.$$

7. A cistern can be filled by two pipes running together in $22\frac{1}{2}$ minutes: the larger pipe would fill the cistern in 24 minutes less than the smaller one: find the time taken by each.

ADVANCED.

1. Derive the formula which gives the sum of n terms of an arithmetical progression. Apply it to find the sum of all the numbers between 200 and 400 which are divisible by 7.

2. Simplify the expression

$$\frac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}} - \frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}} + \frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}.$$

3. How many significant numbers can be formed by using any number of the digits 0, 1, 2, 3, but using each not more than once?

4. Resolve into partial fractions $\frac{x^2-x+1}{(x^2+1)(x-1)^2}$.

5. Show that

$$\log\frac{\sqrt[4]{5}\,\sqrt[10]{2}}{\sqrt[3]{18\sqrt{2}}} = \tfrac{1}{4}\log 10 - \tfrac{13}{20}\log 2 - \tfrac{2}{3}\log 3.$$

6. Find a commensurable root of the equation

$$x^4+4x^3-8x^2-43x-44=0.$$

Then reduce the degree of the equation and find an incommensurable root to 3 decimal places.

COLLEGE OF NEW JERSEY.

JUNE, 1892.

1. Factor (*a*) $1 - 18x - 63x^2$,

(*b*) $(4ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$;

and find the greatest common divisor of

$$a^2 - 5ab + 4b^2 \text{ and } a^3 - 5a^2b + 4b^3.$$

2. Simplify

$$\frac{x^2 - yz}{(x+y)(x+z)} + \frac{y^2 - zx}{(y+z)(y+x)} + \frac{z^2 - xy}{(z+x)(z+y)}.$$

3. Extract the square root of

$$x^{\frac{5}{3}} - 4x^{\frac{4}{3}} + 2x + 4x^{\frac{2}{3}} + x^{\frac{1}{3}}.$$

4. Solve the equations

$$\frac{x}{a} + \frac{y}{b} = c,$$

$$\frac{x}{a'} + \frac{y}{b'} = c' \qquad \text{for } x \text{ and } y.$$

5. Solve (*a*) $\dfrac{5}{4x^2 - 1} - \dfrac{3}{2x + 1} = \dfrac{2}{3}$ for x.

Also (*b*) $(x^2 + x + 1)(x^2 + x + 2) = 12$.

6. Solve (*a*) $x + 4y = 2 - x$,

and $2x^2 + y^2 - xy + 5x + 3y = 34$ for x and y.

Also (*b*) $x + y = 8xy$,

$x^2 + y^2 = 40x^2y^2$.

JUNE, 1893.

1. (*a*) Show that

$$\frac{(n+1)(n+2)(n+3)}{2 \times 3} - 1 = \frac{n}{6}(n^2 + 6n + 11).$$

(*b*) Find the algebraic expression which when divided by $x^2 - 2x + 1$ gives a quotient $x^2 + 2x + 1$ and a remainder $x - 1$.

2. (*a*) Reduce to a common denominator (arranging the terms of the numerator according to ascending powers of x)

$$\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3}.$$

(*b*) Having given that: $A=2$, $B-2A=0$, $C-2B+3A=-3$, $D-2C+3B=-1$, $E-2D+3C=0$; find the values of B, C, D, and E.

3. Solve for x and y $2x-3y+14=0$, $5y-4x=26$.

4. (*a*) Simplify $x^{3p+q}.x^{p-4r}.(x^2)^{q-2r}\div x^{4p-5r}$.

(*b*) Multiply $x^n+x^{\frac{n}{2}}+1$ by $x^{-n}+x^{-\frac{n}{2}}+1$.

5. Solve for x (*a*) $x^2+2a^2=3ax$.

Also (*b*) $x^{\frac{1}{3}}+3x^{-\frac{1}{3}}=4$.

6. Solve the simultaneous equations

$$(a)\ \frac{x}{2}+\frac{y}{5}=5,\qquad \frac{2}{x}+\frac{5}{y}=\frac{5}{6}.$$

Also $$(b)\ x+y=2a,\qquad x^2+y^2=2a^2.$$

SHEFFIELD SCIENTIFIC SCHOOL.

JUNE, 1891.

ALGEBRA TO QUADRATICS.

1. Simplify $a-[2a-b-(3a-2b-\overline{4a-3b})]$.

2. Resolve the following expressions into factors: (*a*) $x^2-(a-b)^2$; (*b*) a^4-1; (*c*) a^3+1; (*d*) $a^2-2ab+b^2-x^2$.

3. Reduce the fraction $\dfrac{12x^2+24x-15}{4x^2+12x+5}$ to its lowest terms.

4. Solve the equation $\dfrac{x+3}{2}-\dfrac{x-2}{3}=\dfrac{3x-5}{12}+\dfrac{1}{4}$.

5. Solve the simultaneous equations

$$\frac{1}{3x}+\frac{1}{5y}=\frac{2}{9}\text{ and }\frac{1}{5x}+\frac{1}{3y}=\frac{1}{4}.$$

6. Expand $(a+b+c)^3$ by the binomial formula.

7. Simplify the following expressions:

(*a*) $(x^6)^{\frac{1}{3}} + 4\sqrt[4]{x^8}$; (*b*) $(x^{\frac{2}{3}} \times x^{\frac{4}{5}})^{\frac{5}{11}}$; (*c*) $\sqrt{75} + \sqrt{48} - \sqrt{243}$
(*d*) $\sqrt{18} \times \sqrt{8}$; (*e*) $\dfrac{x-1}{x+1}\sqrt{\dfrac{x+1}{x-1}}$.

ALGEBRA FROM QUADRATICS.

1. Solve the equation $ax^2 + bx + c = 0$, and point out what relation must exist between the coefficients in order that the roots may be equal.

2. Determine by inspection the roots of the equation

$$x(x+2)(x^2-4) = 0.$$

3. Solve the equation

$$x^2 - x + 5\sqrt{2x^2 - 5x + 6} = \frac{3x + 33}{2}.$$

4. There are 20 things of one kind and 10 of another how many different sets can be made, each containing 3 of the first kind and 2 of the second?

5. Insert 3 arithmetical means between 4 and 20.

6. Expand $\dfrac{1-x}{1+x+x^2}$ into a series by the method of undetermined coefficients.

7. Find the limit of $\dfrac{(x+1)(x^2-3)}{x^4-3x}$ when x increases without limit.

JUNE, 1892.

ALGEBRA TO QUADRATICS.

1. Reduce to their simplest forms the following expressions:

(*a*) $\dfrac{6x^2 - 5x - 6}{4x^3 - 9x}$;

(*b*) $\left(\dfrac{x}{1+x} + \dfrac{1-x}{x}\right) \div \left(\dfrac{x}{1+x} - \dfrac{1-x}{x}\right)$.

2. Solve the equation $\frac{9x+7}{2} - \left(x - \frac{x-2}{7}\right) = 36.$

3. Solve the simultaneous equations

$$3ax - 2by = c,$$
$$a^2x + b^2y = 5bc.$$

4. Prove that if a and b are positive numbers $\frac{a}{b} + \frac{b}{a} > 2.$

5. Write the expansion of $(a - b)^6$ by the binomial formula.

6. Which is greater $\left(\frac{1}{2}\right)^{\frac{1}{2}}$ or $\left(\frac{2}{3}\right)^{\frac{2}{3}}$? Prove your answer.

7. Simplify the following expressions:

(*a*) $(a+b)^{\frac{1}{m}} \times (a+b)^{\frac{1}{n}} \times (a-b)^{\frac{1}{m}} \times (a-b)^{\frac{1}{n}};$

(*b*) $72^{\frac{1}{3}} - 3\left(\frac{1}{3}\right)^{\frac{1}{3}};$ (*c*) $\frac{5}{14}\left(\frac{2}{3}\right)^{\frac{1}{3}} \div \frac{5}{21}\left(\frac{9}{4}\right)^{\frac{1}{3}};$ (*d*) $(ab^2c^m)^{\frac{1}{m}}.$

ALGEBRA FROM QUADRATICS.

1. Solve the equation $9x^2 + 6x = 19.$

2. Determine by inspection the *nature* of the roots of the following equations, that is, whether the roots are real and equal, real and unequal, or imaginary:

(*a*) $2x^2 - 3x + 5 = 0;$ (*b*) $3x^2 - 5x = 2;$ (*c*) $9x^2 - 12x + 4 = 0.$

3. Form a quadratic equation whose second member shall be zero, whose known term in the first member shall be -4, and one of whose roots shall be $-\frac{1}{3}$.

4. Ascertain the square root of $14 + 6\sqrt{5}$.

5. Find the formula for the sum of a decreasing geometric progression of an infinite number of terms.

6. Write the formula expressing the number of combinations of m different things taken n at a time.

7. Expand $\frac{x}{3 - 2x^2}$ into a series of ascending powers of x, by the method of undetermined coefficients. (Four terms will suffice.)

JUNE, 1893.

ALGEBRA TO QUADRATICS.

1. If $a + b + c = 2s$, prove that $a^2 - (b - c)^2$
$= 4(s - b)(s - c)$.

2. Solve the equation $\frac{x - a}{2x - b} - \frac{3x - c}{6x - d} = 0$.

3. Solve the simultaneous equations

$$\frac{1}{2(x+1)} + \frac{4}{3(y+1)} = 5, \quad \frac{1}{x+1} - \frac{1}{3(y+1)} = 1,$$

and verify your solution.

4. Given $2x - 3 < x + 5$ and $11 + 2x < 3x + 5$, to find limits between which x lies.

5. Show that if $a : b = c : d$, then $a + b : a - b = c + d : c - d$.

6. What term in the devleopment of $\left(a + \frac{1}{a}\right)^{2n}$ does not contain a?

7. Simplify the following expressions:

(*a*) $2\sqrt{3} \times 3\sqrt{3}$; (*c*) $(3^{\frac{1}{3}} - 3^{-\frac{1}{3}})^8$; (*e*) $\frac{2}{3}\sqrt{\frac{3}{2}}$;

(*b*) $4\sqrt{8} \div 2\sqrt{2}$; (*d*) $\sqrt{18} - \sqrt{8}$; (*f*) $\frac{a^0\, a^{\frac{1}{2}}}{a^{\frac{1}{3}}}$.

ALGEBRA FROM QUADRATICS.

1. (*a*) Solve the equation $ax^2 + bx + c = 0$.

(*b*) What relation must exist between its coefficients in order (1) that its roots may be imaginary, (2) that they may be real and equal, (3) that they may be real and unequal?

(*c*) If the coefficient a diminishes without limit, what limits, if any, do the roots respectively approach?

2. Make the first members of the following equations perfect squares, without introducing fractions

$$2x^2 - 3x = 2,\ 3x^2 - 8x = -4.$$

3. Solve completely the simultaneous equations $x^2 + xy + y^2 = 19$, $x^2 - xy + y^2 = 7$, and group distinctly the corresponding values of x and y.

4. Convert 3.14159 into a continued fraction, and obtain four convergents. What is the limit of the error in taking the fourth convergent for the value of the decimal?

5. (*a*) Derive the formula for the number of permutations of m things taken n at a time.

(*b*) From 10 different things in how many ways can a selection of 4 be made?

6. (*a*) Write equivalents for the following expressions: $\log_a 1$; $\log_a a$; $\log_a 0$, if $a > 1$; $\log a^2 - \log b^2$; $\log \sqrt{\left(\frac{a^2 - b^2}{c^3}\right)^5}$.

(*b*) Given the mantissa of $\log_{10} 257 = 0.40993$, to find

$$\log_{10} \sqrt{0.257}.$$

7. Given the $\log_a \mathrm{N}$ and $\log_a b$, to find $\log_b \mathrm{N}$.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

JUNE, 1891.

PRELIMINARY.

1. Remove the parentheses from

$$2a - [2b - a - \{b - (a - b) + (b - a) - \{-b - \overline{a - b}\}\}].$$

2. Factor $a^2bc - ac^2d - ab^2d + bcd^2$ and $a^6 - 64$.

3. At what time between 6 and 6.30 o'clock are the hands of a watch at right angles to each other?

4. Solve the equations

$$\frac{1}{x} + \frac{1}{y} = 1,\ \frac{1}{y} + \frac{1}{z} = \frac{3}{2},\ \frac{1}{z} + \frac{1}{x} = 2.$$

5. Extract the square root of $\frac{7}{18}$ to five decimal places.

6. Reduce to its simplest form $\left(\frac{a^{x+y}}{a^y}\right)^x \div \left(\frac{a^y}{a^{y-x}}\right)^{x-y}$.

7. Solve the equation $\frac{\sqrt{x}-3}{\sqrt{x}+7} = \frac{\sqrt{x}-4}{\sqrt{x}+1}$.

FINAL.

1. Solve the equations $x^2 + 3xy = -14$.

$$xy + 4y^2 = 30.$$

2. Solve the equation $x^3 + x^2 - x - 1 = 0$.

3. In an arithmetical progression given $d = 4$, $l = 75$, $n = 19$. Find a and S.

4. Find the sum to infinity of the series

$$-8, -\tfrac{2}{5}, -\tfrac{1}{50}, \ldots$$

5. Find the fifth term of $(x - \sqrt{x})^{18}$.

6. Derive the formula for the sum of a series in geometrical progression.

7. Determine by inspection the sum and product of the roots of the equation $3x^2 - 5x + 4 = 0$.

8. Factor $4 - 32n^2 + 49n^4$.

COMPLETE.

1. Resolve into four factors $(a^2 - b^2 - 9)^2 - 36b^2$.

2. Find the value of $x^2 + xy + y^2$, when $x = \sqrt{3} + \sqrt{2}$, and $y = \sqrt{3} - \sqrt{2}$.

3. Solve $\frac{10}{4x-3} - \frac{5}{2x-1} = \frac{1}{2x+3}$.

4. Extract the square root of $a^4 + 12a^3 - 216a + 324$.

5. Solve $\sqrt[3]{14-x} + \sqrt[3]{14+x} = 4$.

6. Form the quadratic equation whose roots are

$$a + b \pm (a - b)\sqrt{-1}.$$

7. Find the coefficient of x^6 in $(2a^2 - 3x^2)^9$.

8. Find x and y from the proportions

$$2x - 2y : 3y - 5x = 5y : 3x - 6y,$$
$$2x - 5 : 2y = x - 3 : y + 1.$$

JUNE, 1892.

PRELIMINARY.

1. Factor (*a*) $a^4 + a^2b^2 - b^2c^2 - c^4$,
(*b*) $a^2 - 4b^2 - 9c^2 + 12bc$.

2. Find the highest common factor and lowest common multiple of $12x^2 - 29x + 14$ and $18x^2 + 3x - 10$.

3. Extract the square root of

$$8x^6 + \frac{1}{8x^6} - 8x^4 + 2x^2 - \frac{1}{x^2} + 2.$$

4. Reduce to a fraction with rational denominator

$$\frac{\sqrt{3}}{2 - \sqrt{3}} - \frac{2 - \sqrt{-2}}{2 + \sqrt{-2}}.$$

5. Solve $$\frac{\sqrt{3x} - \sqrt{3}}{\sqrt{2x} - \sqrt{2}} = \frac{\sqrt{x} + 3}{\sqrt{x} + 2}.$$

6. What number must be subtracted from r^2 and q^2 in order that the differences may be in the ratio of 6 to -11?

7. Find three numbers, such that the first with half the other two, the second with one-third the other two, and the third with one fourth the other two, shall each equal 34.

8. Find the value of

$$\frac{9^{-2} \div 16^{\frac{7}{4}}}{81^{-\frac{3}{2}} \div 4^{\frac{3}{2}}}.$$

FINAL.

1. Solve $$x^2 + xy + y^2 = 52,$$
$$xy - x^2 = 8.$$

2. What number added to 2, 20, 9, 34, will make the results proportional?

3. Expand $(x^a - x^{-2a})^6$.

4. How many times will a clock strike in a day, the dial being numbered from 1 to 24? Solve by algebraic methods only.

5. Solve $x^4 - \frac{3x^2}{2} + 1 = 0$.

6. Find the value of the recurring decimal .215454 . . .

7. Prove that in a geometrical progression

$$S = \frac{rl - a}{r - 1}.$$

8. If four quantities are in proportion, prove that they will be in proportion by division.

COMPLETE.

1. Find the value of

$$(a^2 - b^2)x^2 - 4abx, \text{ when } x = \frac{a+b}{a-b}.$$

2. Resolve into four factors

$$a^2x^3 + 8a^2y^3 - 4b^2x^3 - 32b^2y^3.$$

3. Solve the equation

$$\frac{x+b-c}{bc} + \frac{x+c-a}{ca} + \frac{x+a-b}{ab} = 1.$$

4. Solve the equation

$$(x^3 + a^3)^{\frac{1}{2}} = x^{\frac{3}{2}} + 4b^{\frac{3}{2}}.$$

5. Simplify $\left(\sqrt{3+\sqrt{5}} - \sqrt{3-\sqrt{5}}\right)^2$.

6. Solve the equation

$$(2x-3)^{-6} + 7(2x-3)^{-3} = 8.$$

7. Find x and y from

$$x^3 - y^3 = 98, \quad x - 2 : y = x - 1 : y + 1.$$

8. What number added to 1, 13, and 73, will give results in geometrical progression?

June, 1893.

PRELIMINARY.

1. Remove the parentheses from the following expression, writing the result as the product of two factors:

$$(x-y)(x^2+xy)-(y-x)y^2+x^2(x-y)-y[x^2-(2x^2-y^2)].$$

2. Simplify $\dfrac{x+y}{2x-2y}+\dfrac{x-y}{2x+2y}-\dfrac{4x^2y^2}{x^4-y^4}$.

3. Find the greatest common factor of

$$x^2+4x+3 \text{ and } 4x^3+4x^2-29x-15.$$

4. Solve the simultaneous equations

$$x+y-3z=8,\quad y+z-3x=-4,\quad z+x-3y=-8.$$

5. Extract the square root of

$$a^2+2a^{\frac{3}{2}}b^{\frac{1}{2}}+3ab+2a^{\frac{1}{2}}b^{\frac{3}{2}}+b^2.$$

6. On a certain street railway two sizes of cars are used. What is the seating capacity of each, if fourteen more persons can be seated in three large cars than in four small cars, and two more persons in two large cars than in three small cars?

FINAL.

1. Solve

$$x+y=x^2,$$
$$3y-x=y^2.$$

2. Expand $(a^{-3}-\frac{1}{3}x^{\frac{1}{2}})^6$.

3. Sum to infinity the geometrical progression

$$5,\ -\frac{1}{2},\ +\frac{1}{20},\ -\frac{1}{200},\ +\text{etc.}$$

4. Insert 6 arithmetical means between 1 and 29.

5. Solve $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}$.

6. For what value of m will the equation

$$2x^2+3mx+2=0, \text{ have}$$

(*a*) equal roots?

(*b*) imaginary roots?

COMPLETE.

1. Resolve into three factors $(x^2 - x)^3 - 8$.

2. Find the greatest common factor of
$$x^3 + 5x^2 + 6x \text{ and } 3x^3 + 7x^2 + 3x + 2.$$

3. Solve the equation
$$\sqrt{x - 4} + \sqrt{x - 11} - \sqrt{2x + 9} = 0.$$

4. Simplify $$\frac{\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} + \dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}}{\dfrac{x^2 + y^2}{x^2 - y^2} + \dfrac{\dfrac{2}{x - y}}{\dfrac{1}{x} + \dfrac{1}{y}}}.$$

5. Solve the simultaneous equations
$$3(x^2 + xy) = 40y, \qquad x - y = 2.$$

6. What is the geometrical mean between
$$2x - 3 \text{ and } 2x^3 + x^2 - 4x - 3?$$

7. A and B start at the same time from the same point in the same direction. A goes at the uniform rate of 60 miles per day; B goes 14 miles the first day, 16 miles the second day, 18 miles the third day, and so on. At the end of fifty days who will be ahead, and by how much?

MISCELLANEOUS QUESTIONS SELECTED FROM EXAMINATION PAPERS

1894–1898.

SHEFFIELD SCIENTIFIC SCHOOL.

LOGARITHMS.

1. Define *a logarithm, its characteristic, its mantissa.*

2. Prove the following:

(*a*) $\log_a a = 1.$

(*b*) $\log_a 1 = 0.$

(*c*) $\log_a \frac{1}{a} = -1.$

(*d*) $\log_a 0 = -\infty$, when $a > 1.$

(*e*) $\log mn = \log m + \log n.$

(*f*) $\log \frac{m}{n} = \log m - \log n.$

(*g*) $\log m^p = p \log m.$

(*h*) $\log m^{\frac{1}{r}} = \frac{\log m}{r}.$

(*i*) $\log_a \mathrm{N} = \log_e \mathrm{N} \cdot \log_a \mathrm{E}.$

(*j*) $\log_{10} e \times \log_e 10 = 1.$

3. (*a*) What is the logarithm of $\frac{1}{8}$ in a system whose base is 2?

(*b*) What is the value of $\frac{1}{3} \log_2 8$? of $\frac{1}{2} \log_3 9 - 2 \log_9 3 + \log_a a$? of $\log_a a + \frac{1}{3} \log_{10} .001 + \frac{2}{3} \log_2 8$? of x in $2^x = 128$?

(*c*) If $\log_{10} 5 = 0.699$, find $\log_5 100.$

4. Show that $\log_{10} 5 = 1 - \log_{10} 2.$

5. Indicate the solution in the following:

$$5^x = 8, \quad 3^{\frac{1}{x}} = 10, \quad a^{mx} b^{nx} = c.$$

6. Given $\log_b a = c$. Express the same relation between the quantities involved, without using logarithmic notation.

7. Find a method of changing the logarithm of a number from one base to another.

8. What is the logarithm of 1 in any system? of the base of any system? of the reciprocal of the base?

9. If the base is greater than 1, for what numbers are the logarithms positive? for what numbers negative?

10. Expand into a form suitable for computation

$$\log\sqrt[5]{\frac{a^3(b-c)^2}{a^2-c^2}},\ \text{also}\ \log\sqrt{\left(\frac{a^2-b^2}{c^3}\right)^5}.$$

Note. — 11–15 are from Harvard Trigonometry papers.

11. If $\log_x 4 = y$ and $\log_x 3 = z$, find $\log_x 2$, $\log_x 12$, and $\log_x 9$. $\text{Log}_4 8 =$ what?

12. Prove $\log\ x^m = m \log_a x$; find by the tables $\log_3 2$.

13. What is the integral part of $\log_2 20$? of $\log_{20} 2$? How are these two logarithms related?

14. Determine the value of x by the tables; $\log_x 5 = \frac{1}{2}$, $\log_{24} 5 = x$.

15. If the base of a system of logarithms is 16, what is the logarithm of 0.015625?

QUADRATIC EQUATIONS.

1. Find *by inspection* the sum and product of the roots in the following:
(*a*) $3x^2 - 5x + 1 = 0$.
(*b*) $ax^2 + bx + c = 0$.

2. For what value of a will the roots be equal in the following:
(*a*) $ax^2 + 6x + 3 = 0$.
(*b*) $2x^2 + ax + \frac{9}{8} = 0$.
(*c*) $3x^2 - 2x + a = 0$.

3. What relation do the roots of the equation $x^2 + px + q = 0$ bear to the coefficients p and q? What will be the value of q if the roots are reciprocals of each other?

4. Solve the equation $ax^2 + bx + c = 0$, and from this result find tests for determining the character of the roots.

5. Determine by inspection the *nature* of the roots in the following equations; that is, whether the roots are real and equal, real and unequal, or imaginary:

$$(a)\ 2x^2 - 3x + 5 = 0.$$
$$(b)\ 3x^2 - 5x = 2.$$
$$(c)\ 9x^2 - 12x + 4 = 0.$$

6. Solve the equations:

$$(a)\ acx^2 - bcx - adx + bd = 0.$$
$$(b)\ a^2x^2 - 2a^3x + a^4 - 1 = 0.$$
$$(c)\ \frac{x}{a^2b(x + c)} = \frac{x + c}{d^2bx}.$$
$$(d)\ \begin{cases} ay^2 + bxy = b. \\ bx^2 + axy = a. \end{cases} \quad \text{(find } x \text{ and } y.\text{)}$$

7. Determine by inspection the roots of $x^2 - 2ax = (b + a)(b - a)$.

8. Construct the equation whose roots are

$$\frac{1 + \sqrt{3}}{2},\ \frac{1 - \sqrt{3}}{2}.$$

9. Solve the simultaneous equations $\begin{cases} x^2 + xy = 12. \\ xy - 2y^2 = 1. \end{cases}$

10. Solve $\sqrt{x^2 - a^2} = 2 - \dfrac{1}{\sqrt{x^2 - a^2}}.$

MISCELLANEOUS.

1. Find two geometrical means between p and q.

2. Find the sum of n terms of the series

$$1 + \frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} + \dots$$

3. Find the value of the recurring decimal 0.43232 . . .

4. Find the limit of the sum of an infinite number of terms of the series 9, − 6, + 4, − . . .

FROM MASSACHUSETTS INSTITUTE OF TECHNOLOGY PAPERS.

5. Show that $2a^2(a+3b)$, $(a+b)^3$, $2b^2(b+3a)$ are in Arithmetical Progression.

6. If a, b, c, are in Geometrical Progression, show that $\frac{a}{a+b}$, $\frac{1}{2}$, $\frac{c}{b+c}$, are in Arithmetical Progression.

7. If $a+b:b+c=c+d:d+a$, prove $a=c$, or $a+b+c+d=0$.

8. If $x-y$ is a mean proportional between y and $y+z-2x$, show that x is a mean proportional between y and z.

9. Show that if b is a mean proportional between a and c,

$$(a^2+b^2)(b^2+c^2)=(ab+bc)^2.$$

10. Solve (a) $4x^{-\frac{4}{3}}-5x^{-\frac{2}{3}}+1=0$.

(b) $10x^{\frac{5}{12}}=3(x^{\frac{1}{2}}+x^{\frac{1}{3}})$.

11. Express as one term $3^{\frac{4}{3}}+\sqrt[3]{24}-\sqrt[3]{9}$.

12. Find the value to four decimal places of $\frac{1}{\sqrt{47}}$ by the Binomial Theorem.

ENTRANCE EXAMINATIONS.

JUNE, 1899.

HARVARD UNIVERSITY.

1. Solve the equations $\frac{4}{x} + \frac{7}{y} = 8\frac{a}{xy}$,

$$x^2 + y^2 = a^2,$$

finding all the sets of values of x and y that satisfy both equations.

2. Two persons, A and B, run a race to go five times round a certain course. When A has gone three laps, B is 150 yards behind him. A then slackens speed and goes at B's rate, while B quickens his rate and goes at A's first rate. A wins by 30 yards. Find the length of the course, and compare the original speeds of A and B.

3. Solve the equation $a^2x - 2b^2 = ab\frac{x^2+1}{x+1}$, reducing the answers to their lowest terms.

4. Find the Greatest Common Divisor and the Least Common Multiple of the expressions

$$x^3y^3 + y^6, \qquad x^3 - x^2y + xy^2.$$

5. Reduce the expression $\frac{b+a}{b-a} - 2\left(\frac{b}{a} - \frac{b}{a-b}\right)$, to a single fraction in its lowest terms.

6. Reduce the expression $\frac{\sqrt[3]{a^2x^{-1}}}{\sqrt[6]{ab^3x}}\sqrt{ab^2x^3}$, to its simplest form, having only one radical sign, and free from negative or fractional exponents.

YALE COLLEGE.

A.

1. Solve $\frac{2}{x} + \frac{3}{y} = 4.$

$\frac{1}{x} + \frac{7}{y} = 6.$

2. Factor $x^2 + 7x - 18.$
Expand and factor $(2x - 5y)^2 - (x - 2y)^2.$

3. Simplify $7\sqrt[3]{54} + \sqrt[6]{256} + \sqrt[3]{432}.$

4. Simplify $\frac{a^{\frac{3}{5}} c^{-1} \sqrt{b}}{\sqrt[4]{b^3}} \times a^{-2} \sqrt[5]{ab^5}.$

5. Find the square root of

$$\frac{a^2x^2}{9} + \frac{b^2}{25} + \frac{9x^4}{4} - ax^3 - \frac{3bx^2}{5} + \frac{2abx}{15}.$$

6. A and B buy stock: A buying twice as much as B. If A had paid $1000 more and B $1000 less, A would have paid three times as much as B. How much money did each invest?

B.

1. Find 3 numbers in the ratio of 1 : 2 : 3, such that the sum of their squares is 350.

2. Solve $9x - 3x^2 + 4\sqrt{x^2 - 3x + 5} = 11.$

3. A man deposits in the bank 1 cent the first day, 2 cents the next day, 4 cents the third day, and so on for sixteen days: find the whole amount deposited.

4. If $\frac{x^2 + 2x + 5}{x(x + 3)(x + 4)} = \frac{A}{x} + \frac{B}{x + 3} + \frac{C}{x + 4}$, find the values of A, B, and C by the method of undetermined coefficients.

5. Expand by the binomial theorem $(x - 2a)^{-20}$ to four terms.

SHEFFIELD SCIENTIFIC SCHOOL.

To Quadratics.

1. Find the highest common factor of

$$2x^3+4x^2+2x \text{ and } 3x^3+9x^2+6x.$$

2. Find the value in its simplest form of

$$\left(\frac{2}{x}-\frac{1}{a+x}+\frac{1}{a-x}\right)\div\left(\frac{a+x}{a-x}-\frac{a-x}{a+x}\right).$$

3. Solve the equation $\frac{5x-1}{8}-\frac{3x-2}{7}=\frac{5-x}{4}$.

4. Solve the simultaneous equations

$$\begin{aligned}2x+3y+4z&=20.\\3x+4y+5z&=26.\\4x+5y+6z&=32.\end{aligned}$$

5. A has a hours to spare for an outing. How far can he ride with a friend at the rate of b miles an hour, and just consume the time in walking back at the rate of c miles an hour?

6. Extract the square root of $x^4-4x^3+8x^2-8x+4$.

7. Simplify the following expressions:

(*a*) $\sqrt{a^2-x^2}+\frac{x^2}{\sqrt{a^2-x^2}}$; (*b*) $a\sqrt[3]{a^3d}+b\sqrt[3]{b^3d}-c\sqrt{c^2d}$;

(*c*) $2\sqrt[3]{3}\times 2\sqrt{2}$; (*d*) $\left(\frac{64a^3}{27b^{-3}}\right)^{-\frac{1}{3}}$.

From Quadratics.

1. (*a*) Determine by inspection the roots of the equation

$$x^2-110x+1000=0, \text{ and}$$

(*b*) Resolve the first member into factors — stating, in each case, the theorem you use.

2. (*a*) Solve the equation $\frac{x+a}{x+b}+\frac{x+b}{x+a}=1$, and verify your solution for $a=b$.

(*b*) Show that the roots of the equation are imaginary for $a \lessgtr b$.

3. Solve the equation $\frac{1}{x^2-3x+1}+\frac{2}{x^2-3x+2}=\frac{8}{15}$, using an auxiliary quadratic.

4. Given $\frac{a}{b}=\frac{c}{d}$ to show that $\frac{a}{b}=\frac{c}{d}=\frac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}$.

5. How many different three-figure integer numbers can be expressed by the nine digits, without repeating any figure in any one number.

6. Find the limit of the sum of the series

$$5+\frac{5}{10}+\frac{5}{100}+\cdots\cdots$$

as the number of terms increases without limit.

7. (*a*) Give the values of the following expressions:

$$\log_5 125;\ \log_5 5;\ \log_5 1.$$

(*b*) Given $\log\,(x^2y^3)=a$ and $\log\frac{x^2}{y^2}=b$, to find $\log x$ and $\log y$.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

1. Simplify

$$\frac{x^2+y^2}{x^2-y^2}+\frac{2x}{x+y}\left\{\frac{xy-x^2}{(x-y)^2}+\frac{x+y}{x-y}\right\}$$

2. Factor the expressions:

$$x^6-64;\ x^4-10x^2+9.$$

3. Simplify

$$\frac{a^{\frac{n+1}{n}}+b\,(ab^{\frac{1-n}{n}}+a^{\frac{1}{n}})+b^{\frac{n+1}{n}}}{a^2-b^2}.$$

4. Solve the equations:

$$\begin{aligned} x + 2y + 3z &= 17. \\ 2x + 3y + z &= 12. \\ 3x + y + 2z &= 13. \end{aligned}$$

5. State fully one method for the solution of quadratic equations.

6. Solve the equation $\sqrt{x^2 - 4} + \sqrt{x^2 + 5} = 3$.

7. Solve the equations $\begin{cases} x + y = 6 \\ 4(x - y) = xy. \end{cases}$

8. Find the limit of the sum of the infinite series

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} + \cdots\cdots$$

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1) $1+x^8+x^{16}$

2) $(x+y)^{\frac{2}{3}}-(x+y)^{\frac{1}{3}}z^{\frac{1}{3}}+z^{\frac{2}{3}}$

3) $\begin{cases}(3x+2)(3x+1)\\(x-1)^3\,(x+1)^3\end{cases}$

3) $\begin{cases}\dfrac{ab-4b^2}{a-3b}\\5,\ -\dfrac{32}{5}\end{cases}$

4) a, a

3) $1,\ -3\pm2\sqrt{2}$

6) $\dfrac{3+\sqrt{-3}+\sqrt{6}}{6}$

7) $1120x^8$

5)

6) $0\quad a\pm\sqrt{3}$

7) $\begin{cases}x=-14\\y=5\end{cases}$ $\begin{cases}x=14\\y=-5\end{cases}$ $\begin{cases}x=-2\\y=3\end{cases}$ $\begin{cases}x=2\\y=-3\end{cases}$

8) 1024

9) $x^{30}-20x^{27}ay+180x^{24}a^2y^2$
$-960x^{21}a^3y^3+$

June 1893

I. $(x+3)(x-2)(x^2-x+6)$

$(m-2n)(m+2n)(m^2-2mn+4n^2)(m^2+2mn+4n^2)$.

II $\dfrac{(x^2+3x-16)(x+3)}{(x+4)(x-3)(x+1)}$

III $\dfrac{2mnRP}{nRP+mRP+mnR+mnP}$

IV $xy+y^2\sqrt{-z}-xz\sqrt{-x}+yz\sqrt{xz}$

V $-(a+6)\pm\sqrt{a^2-6a+36}$

VI $x^{\frac{3}{2}}y^{-\frac{1}{3}} - 2+x^{-\frac{3}{2}}y^{\frac{1}{3}}$

VII $\dfrac{a+1}{2a+1}$

VIII $-\frac{7}{8}a^{-\frac{9}{2}}b^5$